

BASICS

Unit 1

- A vector is a quantity that has both magnitude and direction.
- A vector quantities include velocity, force, displacement and electric field intensity.
- A vector represented by a letter with an arrow on top of it. Such as $\vec{A}$ and $\vec{B}$ or by a letter in bold face type such as $\mathbf{A}$ and $\mathbf{B}$.
- A scalar is quantity that has only Magnitude quantity such as time, mass, distance, t.c, electric potential and population.
- A scalar is represented simply by a letter. e.g. A, B, V
- Field ÷ is a function that specifies a particular quantity every where in a region.
- If the quantity is scalar, the field is said to be a scalar field.
- e.g. of scalar fields are → t.c distribution in building
→ sound intensity in theater
- e.g. of vector fields are → the gravitational force on body in space.

Unit Vector ÷ A unit vector $\mathbf{a}_A$ along $\mathbf{A}$ is defined as a vector whose magnitude is unity (i.e. 1) and its direction is along $\mathbf{A}$, that is

$$\mathbf{a}_A = \frac{\mathbf{A}}{|\mathbf{A}|}$$

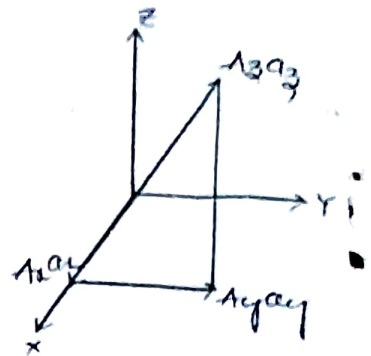
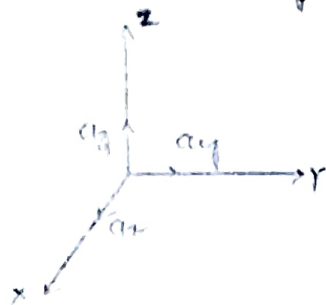
→ A vector $\mathbf{A}$ in cartesian co-ordinates may be represented as

$$\mathbf{A} = A_x \mathbf{a}_x + A_y \mathbf{a}_y + A_z \mathbf{a}_z$$

where A_x, A_y and A_z are called the components of A

in the x, y and z directions respectively.

a_x, a_y and a_z are unit vectors in the x, y and z directions respectively.



$\therefore$ Unit Vectors a_x, a_y & a_z

$\therefore$ components of A along a_x, a_y, a_z

$\therefore$ Magnitude of A is given by $A = \sqrt{A_x^2 + A_y^2 + A_z^2}$

Unit vector along A is given by

$$a_A = \frac{A_x a_x + A_y a_y + A_z a_z}{\sqrt{A_x^2 + A_y^2 + A_z^2}}$$

Vector Addition and Subtraction

Two Vectors A and B can be added together to give another Vector C that is

$$C = A + B$$

The Vector addition is carried out component by component

thus, If $A = (A_x, A_y, A_z)$ and $B = (B_x, B_y, B_z)$

$$C = (A_x + B_x) a_x + (A_y + B_y) a_y + (A_z + B_z) a_z$$

Vector subtraction is similarly carried out

$$D = A - B$$

$$D = (A_x - B_x) a_x + (A_y - B_y) a_y + (A_z - B_z) a_z$$

Prob: If $A = 10a_x - 4a_y + 6a_z$ and $B = 2a_x + a_y$ find

- the component of A along a_y
- the Magnitude of $3A - B$
- unit vector along $A + 2B$.

Solu: a) The component of A along a_y is $A_y = -4$

$$\begin{aligned} b) \quad 3A - B &= 3(10, -4, 6) - (2, 1, 0) \\ &= (28, -13, 18) \end{aligned}$$

$$|3A - B| = \sqrt{(28)^2 + (-13)^2 + (18)^2} = 35.74$$

$$c) \quad \text{Let } C = A + 2B \Rightarrow (14, -2, 6)$$

Unit vector along C

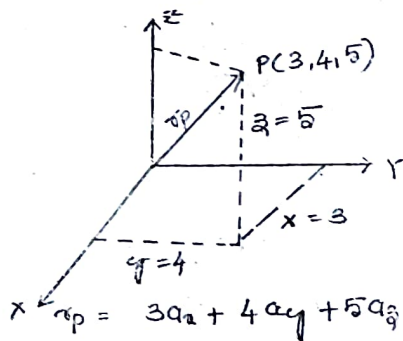
$$a_c = \frac{C}{|C|} = \frac{14a_x - 2a_y + 6a_z}{\sqrt{(14)^2 + (-2)^2 + (6)^2}}$$

$$a_c = 0.911a_x - 0.130a_y + 0.39a_z$$

Position and Distance vectors :

- A point P in cartesian co-ordinates may be represented by (x, y, z)

Position vector (or) radius vector : r_p of point P is as the direct distance from the origin O to P .



- Distance vector : is the displacement from one point to another.

- If 2 points P and Q are given by (x_p, y_p, z_p) and (x_q, y_q, z_q) the distance vector (or) separation vector is the displacement from P to Q .

$$= (x_Q - x_P)a_x + (y_Q - y_P)a_y + (z_Q - z_P)a_z$$

Prob: Points P and Q are located at (0, 2, 4) and (-3, 1, 5)

- The position vector P
- The distance vector from P to Q
- The distance b/w P and Q

Soln:

a) $r_P = 0.a_x + 2.a_y + 4.a_z$

b) $r_{PQ} = r_Q - r_P = (-3, 1, 5) - (0, 2, 4) \Rightarrow -3a_x - a_y + a_z$

c) Since r_{PQ} is the distance vector from P to Q, the distance b/w P and Q is the magnitude of this vector.

$$d = \|r_{PQ}\| = \sqrt{(x_Q - x_P)^2 + (y_Q - y_P)^2 + (z_Q - z_P)^2}$$

$$d = 3.317$$

Vector Multiplication:

When 2 vectors A & B are multiplied. The result is either a scalar or a vector depending on how they are multiplied. There are 2 types of vector multiplications.

- Scalar (or) dot Product : $A \cdot B$
- Vector (or) Cross product : $A \times B$.

Multiplication of 3 vectors A, B and C can result

- Scalar triple product : $A \cdot (B \times C)$
- Vector triple product : $A \times (B \times C)$

1. Dot Product:

The dot product of 2 vectors A & B written as $A \cdot B$ is defined as the product of the magnitudes of A and B and the cosine of the angle between them.

$$A \cdot B = AB \cos \theta_{AB}$$

Where θ_{AB} is the smaller angle b/w A and B .

If $A = (A_x, A_y, A_z)$ and $B = (B_x, B_y, B_z)$ then

$$A \cdot B = A_x B_x + A_y B_y + A_z B_z$$

Two vectors A and B are said to be orthogonal (or) perpendicular with each other if $A \cdot B = 0$

Dot product obeys the following.

i) Commutative Law :

$$A \cdot B = B \cdot A$$

ii) Distributive Law :

$$A \cdot (B + C) = A \cdot B + A \cdot C$$

$$A \cdot A = |A|^2 = A^2$$

$$\text{iii) } a_x \cdot a_y = a_y \cdot a_z = a_z \cdot a_x = 0$$

$$a_x \cdot a_x = a_y \cdot a_y = a_z \cdot a_z = 1$$

2. Cross Product :-

The cross Product of 2 vectors A & B written as $A \times B$.

The product of magnitude of A & B and sine of the angle between them.

$$A \times B = AB \sin \theta_{AB}$$

If $A = (A_x, A_y, A_z)$ and $B = (B_x, B_y, B_z)$

$$A \times B = \begin{vmatrix} a_x & a_y & a_z \\ A_x & A_y & A_z \\ B_x & B_y & B_z \end{vmatrix}$$

$$A \times B = (A_y B_z - A_z B_y) a_x + (A_z B_x - A_x B_z) a_y + (A_x B_y - A_y B_x) a_z$$

cross Product has the following rules

i) It is not commutative

$$A \times B \neq B \times A$$

It is not anti commutative

$$A \times B = -B \times A$$

$$A \times (B \times C) \neq (A \times B) \times C$$

iii) It is distributive

$$A \times (B + C) = A \times B + A \times C$$

iv)

$$A \times A = 0$$

$$a_x \times a_y = a_z$$

$$a_y \times a_z = a_x$$

$$a_z \times a_x = a_y$$

3. Scalar Triple Product ÷

Given 3 vectors A, B and C , we define the scalar triple product as

$$A \cdot (B \times C) = B \cdot (C \times A) = C \cdot (A \times B)$$

4. Vector Triple product ÷

For vectors A, B and C , we define the vector triple product as

$$A \times (B \times C) = B(A \cdot C) - C(A \cdot B)$$

Prob ÷ Given Vectors $A = 3a_x + 4a_y + a_z$ and $B = 2a_y - 5a_z$ find the angle b/w A and B .

soln ÷ The angle θ_{AB} can be found by using either dot product or cross product.

$$A \cdot B = (3, 4, 1) \cdot (0, 2, -5)$$

$$A \cdot B = 0 + 8 - 5 = 3$$

$$|A| = \sqrt{3^2 + 4^2 + 1} = \sqrt{26}$$

$$|B| = \sqrt{0 + 2^2 + 5^2} = \sqrt{29}$$

$$\cos \theta_{AB} = \frac{A \cdot B}{|A||B|} = \frac{3}{\sqrt{26}\sqrt{29}} = 0.1092$$

$$\theta_{AB} = \cos^{-1}(0.1092) = 83.73^\circ$$

$$A \times B = \begin{vmatrix} a_x & a_y & a_z \\ 3 & 4 & 1 \\ 0 & 2 & -5 \end{vmatrix}$$

$$A \times B = -22a_x + 15a_y + 6a_z$$

$$|A \times B| = \sqrt{(22)^2 + (15)^2 + (6)^2} = \sqrt{745}$$

$$\sin \theta_{AB} = \frac{|A \times B|}{|A||B|} = \frac{\sqrt{745}}{\sqrt{(26)(29)}} = 0.994$$

$$\theta_{AB} = \sin^{-1}(0.994) = 83.93^\circ$$

→ A point or vector can be represented in any curvilinear co-ordinate system, which may be orthogonal or non-orthogonal.

→ An orthogonal system is one which the co-ordinates are mutually perpendicular.

→ e.g of orthogonal system include the cartesian (or) rectangular, the circular cylindrical, the spherical, the elliptic cylindrical e.t.c.

→ Non-orthogonal system are hard to work and they are of little or no practical use.

Co-ordinate Systems And Transformation ÷

1. cartesian (or) rectangular co-ordinates (x, y, z)
2. circular cylindrical co-ordinates (ρ, ϕ, z)
3. spherical co-ordinates (r, θ, ϕ)

1. Cartesian Co-ordinates (x, y, z) ÷

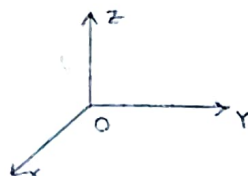
A point P in cartesian co-ordinates is represented as (x, y, z)

The ranges of the co-ordinate variables x, y and z are

$$-\infty < x < \infty$$

$$-\infty < y < \infty$$

$$-\infty < z < \infty$$

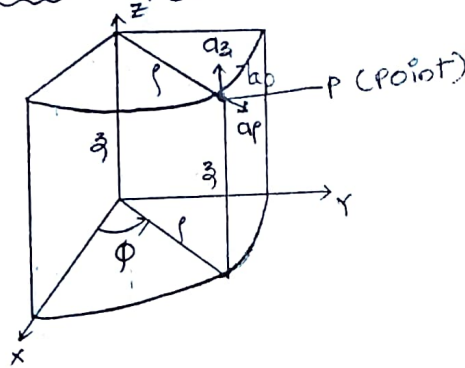


A vector $\vec{A}$ in Cartesian co-ordinates can be written as

$$(\vec{A}_x, \vec{A}_y, \vec{A}_z) \text{ or } A_x \vec{a}_x + A_y \vec{a}_y + A_z \vec{a}_z$$

where $\vec{a}_x, \vec{a}_y, \vec{a}_z$ are unit vectors along x, y and z - directions.

2. Circular (or) cylindrical co-ordinates $(\rho, \phi, z) :$



$\div$ point P and unit vector in the cylindrical co-ordinate system

$\rightarrow$ A point P in cylindrical co-ordinates is represented as ρ, ϕ, z

$\rightarrow \rho$ is the radius of the cylinder passing through P (or) the radial distance from the z -axis; ϕ called as the azimuthal angle, is measured from the x -axis in the xy -plane and z is the same as in the Cartesian system

$\rightarrow$ The ranges of the variables are

$$0 \leq \rho < \infty$$

$$0 \leq \phi < 2\pi$$

$$-\infty \leq z < \infty$$

$\rightarrow$ A vector $\vec{A}$ in cylindrical co-ordinates can be written as

$$(\vec{A}_\rho, \vec{A}_\phi, \vec{A}_z) \text{ or } A_\rho \vec{a}_\rho + A_\phi \vec{a}_\phi + A_z \vec{a}_z \text{ where } \vec{a}_\rho, \vec{a}_\phi, \vec{a}_z \text{ are}$$

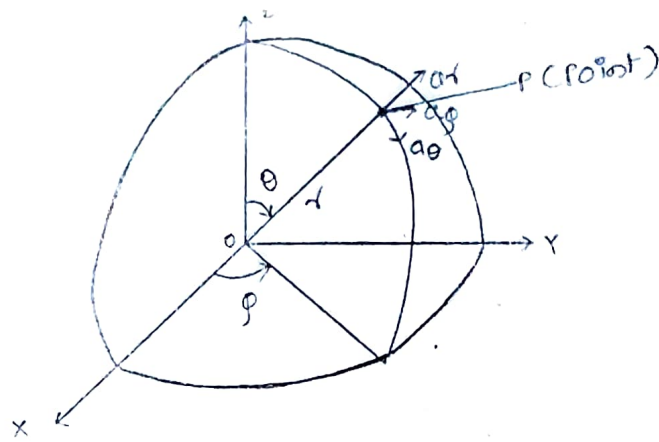
unit vectors in the ρ, ϕ , and z directions.

The Magnitude of $\vec{A}$ is $|\vec{A}| = \sqrt{A_\rho^2 + A_\phi^2 + A_z^2}$

$\rightarrow$ Note that the unit vectors $\vec{a}_\rho, \vec{a}_\phi$ and $\vec{a}_z$ are mutually perpendicular. Because co-ordinate system is orthogonal.

$$\vec{a}_\rho \cdot \vec{a}_\rho = \vec{a}_\phi \cdot \vec{a}_\phi = \vec{a}_z \cdot \vec{a}_z = 1$$

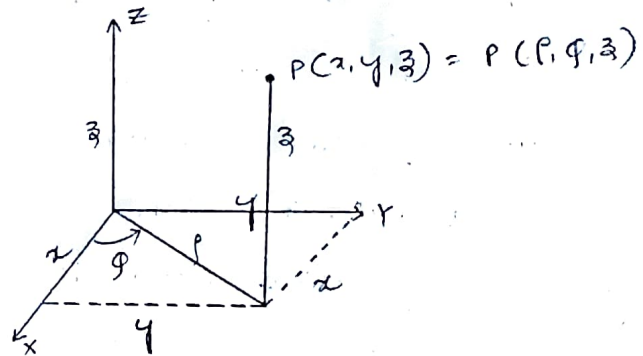
$$\vec{a}_\rho \cdot \vec{a}_\phi = \vec{a}_\phi \cdot \vec{a}_z = \vec{a}_z \cdot \vec{a}_\rho = 0$$



÷ point P and unit vectors in spherical co-ordinates ÷

Transformation of Cartesian co-ordinates to circular co-ordinates

$$(x, y, z) \longrightarrow (r, \phi, z)$$



$$x^2 = r^2 \cos^2 \phi$$

$$y^2 = r^2 \sin^2 \phi$$

$$r = \sqrt{x^2 + y^2}$$

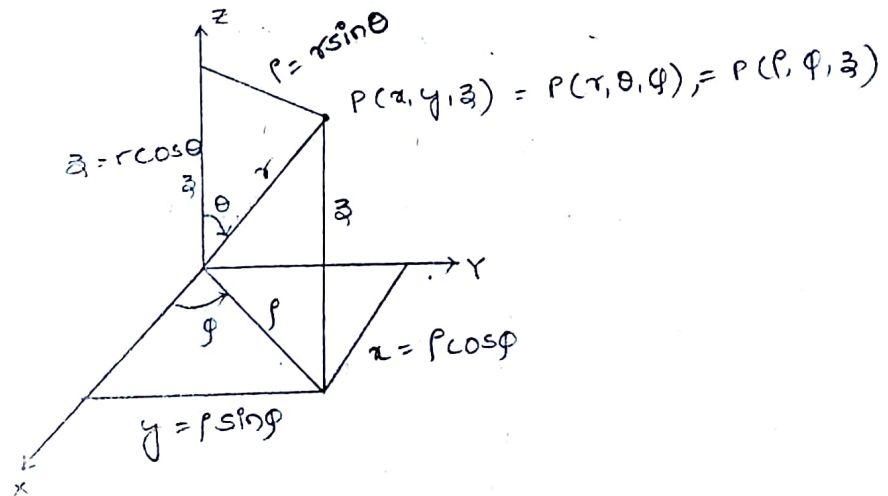
$$\cos \phi = \frac{x}{r}$$

$$\sin \phi = \frac{y}{r}$$

$$r = \sqrt{x^2 + y^2}; \quad \tan \phi = \frac{\sin \phi}{\cos \phi} \rightarrow \tan \phi = \frac{y}{x} \rightarrow \phi = \tan^{-1}\left(\frac{y}{x}\right)$$

$$z = z$$

Transformation of Cartesian to Spherical co-ordinate system ÷



$$a_\phi \times a_\theta = a_z$$

$$a_\theta \times a_z = a_\phi$$

$$a_z \times a_\phi = a_\theta$$

→ a_ϕ in the direction of increasing ϕ , a_θ in the direction of increasing θ and a_z in the +ve z -direction.

3. Spherical co-ordinates (r, θ, ϕ)

→ A point P in spherical co-ordinates represented as $a(r, \theta, \phi)$

→ r is the distance from the origin to point P .

→ θ (called the colatitude) is the angle b/w the z -axis and the position vector of P .

→ ϕ is measured from the x -axis (the same azimuthal angle in cylindrical co-ordinates)

→ The ranges of the variables are

$$0 \leq r < \infty$$

$$0 \leq \theta \leq \pi$$

$$0 \leq \phi \leq 2\pi$$

→ A vector A in spherical co-ordinates may be written as (A_r, A_θ, A_ϕ) or $(A_r a_r + A_\theta a_\theta + A_\phi a_\phi)$. where a_r, a_θ, a_ϕ are unit vectors along the r, θ, ϕ directions. The Magnitude of

$$|A| = \sqrt{A_r^2 + A_\theta^2 + A_\phi^2}$$

$$a_r \cdot a_r = a_\theta \cdot a_\theta = a_\phi \cdot a_\phi = 1$$

$$a_r \cdot a_\theta = a_\theta \cdot a_\phi = a_\phi \cdot a_r = 0$$

$$a_r \times a_\theta = a_\phi$$

$$a_\theta \times a_\phi = a_r$$

$$a_\phi \times a_r = a_\theta$$

$$\begin{bmatrix} A_x \\ A_y \\ A_z \end{bmatrix} = \begin{bmatrix} a_x \cdot a_r & a_x \cdot a_\theta & a_x \cdot a_\phi \\ a_y \cdot a_r & a_y \cdot a_\theta & a_y \cdot a_\phi \\ a_z \cdot a_r & a_z \cdot a_\theta & a_z \cdot a_\phi \end{bmatrix} \begin{bmatrix} A_r \\ A_\theta \\ A_\phi \end{bmatrix}$$

Unit Vectors a_x, a_y, a_z and a_r, a_θ, a_ϕ are related following

$$a_x = \sin\theta \cos\phi a_r + \cos\theta \cos\phi a_\theta - \sin\phi a_\phi$$

$$a_y = \sin\theta \sin\phi a_r + \cos\theta \sin\phi a_\theta + \cos\phi a_\phi$$

$$a_z = \cos\theta a_r - \sin\theta a_\theta$$

(or)

$$a_r = \sin\theta \cos\phi a_x + \sin\theta \sin\phi a_y + \cos\theta a_z$$

$$a_\theta = \cos\theta \cos\phi a_x + \cos\theta \sin\phi a_y - \sin\theta a_z$$

$$a_\phi = -\sin\phi a_x + \cos\phi a_y$$

substitute a_x, a_y and a_z are in eqn no (1)

$$A = A_x (\sin\theta \cos\phi a_r + \cos\theta \cos\phi a_\theta - \sin\phi a_\phi) + A_y (\sin\theta \sin\phi a_r + \cos\theta \sin\phi a_\theta + \cos\phi a_\phi) + A_z (\cos\theta a_r - \sin\theta a_\theta)$$

$$A = a_r (A_x \sin\theta \cos\phi + A_y \sin\theta \sin\phi + A_z \cos\theta) + a_\theta (A_x \cos\theta \cos\phi + A_y \cos\theta \sin\phi - A_z \sin\theta) + a_\phi (-A_x \sin\phi + A_y \cos\phi)$$

$$A_r = A_x \sin\theta \cos\phi + A_y \sin\theta \sin\phi + A_z \cos\theta$$

$$A_\theta = A_x \cos\theta \cos\phi + A_y \cos\theta \sin\phi - A_z \sin\theta$$

$$A_\phi = -A_x \sin\phi + A_y \cos\phi$$

In matrix form $(A_x, A_y, A_z) \rightarrow (A_r, A_\theta, A_\phi)$ vector transformation is performed according to

$$\begin{bmatrix} A_r \\ A_\theta \\ A_\phi \end{bmatrix} = \begin{bmatrix} \sin\theta \cos\phi & \sin\theta \sin\phi & \cos\theta \\ \cos\theta \cos\phi & \cos\theta \sin\phi & -\sin\theta \\ -\sin\phi & \cos\phi & 0 \end{bmatrix} \begin{bmatrix} A_x \\ A_y \\ A_z \end{bmatrix}$$

$$a_x = \cos\phi a_p - \sin\phi a_q$$

$$a_y = \sin\phi a_p + \cos\phi a_q$$

$$a_z = a_3$$

(or)

$$a_p = \cos\phi a_x + \sin\phi a_y$$

$$a_q = -\sin\phi a_x + \cos\phi a_y$$

$$a_3 = a_z$$

Substitute a_x , a_y and a_3 in eqⁿ no ① we get

$$A = A_x (\cos\phi a_p - \sin\phi a_q) + A_y (\sin\phi a_p + \cos\phi a_q) + A_3 a_3$$

$$A = a_p (-A_x \cos\phi + A_y \sin\phi) + a_q (A_x \sin\phi + A_y \cos\phi) + A_3 a_3$$

(or) $A = A_p a_p + A_q a_q + A_3 a_3$

$$A_p = -A_x \cos\phi + A_y \sin\phi$$

$$A_q = A_x \sin\phi + A_y \cos\phi$$

$$A_3 = A_z$$

In matrix form the transformation of vector A form

(A_x, A_y, A_z) to (A_p, A_q, A_3)

$$\begin{bmatrix} A_p \\ A_q \\ A_3 \end{bmatrix} = \begin{bmatrix} \cos\phi & \sin\phi & 0 \\ -\sin\phi & \cos\phi & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} A_x \\ A_y \\ A_z \end{bmatrix}$$

The inverse of the transformation $(A_p, A_q, A_3) \rightarrow (A_x, A_y, A_z)$

$$\begin{bmatrix} A_x \\ A_y \\ A_z \end{bmatrix} = \begin{bmatrix} \cos\phi & \sin\phi & 0 \\ -\sin\phi & \cos\phi & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} A_p \\ A_q \\ A_3 \end{bmatrix}$$

$$\begin{bmatrix} A_x \\ A_y \\ A_z \end{bmatrix} = \begin{bmatrix} \cos\phi & -\sin\phi & 0 \\ \sin\phi & \cos\phi & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} A_p \\ A_q \\ A_3 \end{bmatrix}$$

Vector Transformation :-

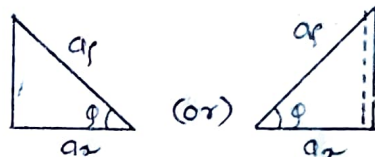
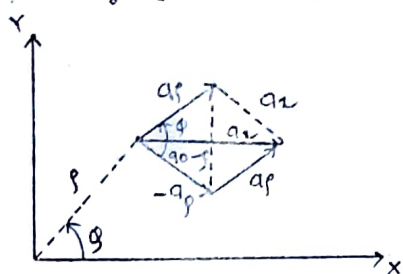
→ A Vector $\vec{A}$ in Cartesian co-ordinates can be written as

$$\vec{A} = A_x \hat{a}_x + A_y \hat{a}_y + A_z \hat{a}_z \quad \text{--- (1)}$$

→ A Vector $\vec{A}$ in Circular cylindrical co-ordinates can be written as $\vec{A} = A_r \hat{a}_r + A_\phi \hat{a}_\phi + A_z \hat{a}_z$

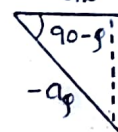
→ The relationships between (A_x, A_y, A_z) and (A_r, A_ϕ, A_z)

:- Unit Vector Transformation :-



$$\cos \phi = \frac{a_x}{a_r}$$

$$a_x = a_r \cos \phi$$



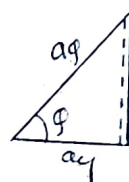
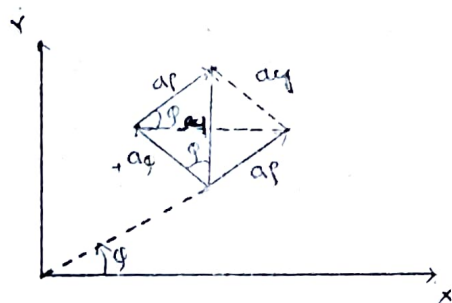
$$\cos(90 - \phi) = \frac{a_x}{-a_y}$$

$$\sin \phi = \frac{a_x}{-a_y}$$

$$a_x = -a_y \sin \phi$$

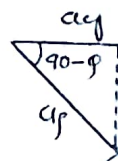
:- cylindrical components of a_x :-

$$a_x = a_r \cos \phi - a_\phi \sin \phi$$



$$\cos \phi = \frac{a_y}{a_\phi}$$

$$a_y = a_\phi \cos \phi$$



$$\cos(90 - \phi) = \frac{a_y}{a_\phi}$$

$$\sin \phi = \frac{a_y}{a_\phi}$$

$$a_y = a_\phi \sin \phi$$

:- cylindrical components of a_y :-

$$a_y = a_r \sin \phi + a_\phi \cos \phi$$

$$x = r \cos \phi \quad \therefore r = \frac{x}{\cos \phi}$$

$$x = r \sin \theta \cos \phi$$

$$y = r \sin \phi \quad \therefore r = \frac{y}{\sin \phi}$$

$$y = r \sin \theta \sin \phi$$

$$z = r \cos \theta$$

$$\sin \theta = \frac{r}{z}$$

$$r = z \sin \theta$$

$$\cos \theta = \frac{r}{z}$$

$$r^2 = x^2 + y^2 + z^2$$

$$r = \sqrt{x^2 + y^2 + z^2}$$

$$\tan \theta = \frac{r \sin \theta}{r \cos \theta} = \frac{r}{z}$$

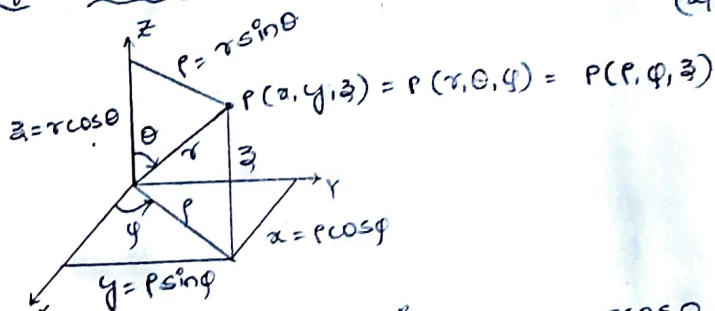
$$\tan \theta = \frac{\sqrt{x^2 + y^2}}{z}$$

$$\therefore r = \sqrt{x^2 + y^2}$$

$$\tan \phi = \left(\frac{y}{x} \right)$$

$$\phi = \tan^{-1} \left(\frac{y}{x} \right)$$

Transformation of spherical to cartesian co-ordinate system $(x, y, z) \rightarrow (r, \theta, \phi)$



$$x = \underbrace{r \sin \theta}_{r} \cos \phi ; y = \underbrace{r \sin \theta}_{r} \sin \phi ; z = r \cos \theta$$

$$[\because r = r \sin \theta]$$

Prob: Given Point $P(-2, 6, 3)$ express P in cylindrical & spherical co-ordinate system.

Soln: at $P: x = -2, y = 6, z = 3$

$$r = \sqrt{x^2 + y^2} = 6.32 ; \phi = \tan^{-1} \left(\frac{y}{x} \right) = 108.43^\circ ; z = 3$$

$$r = \sqrt{x^2 + y^2 + z^2} = 7, \theta = \tan^{-1} \frac{\sqrt{x^2 + y^2}}{z} \rightarrow 64.62^\circ$$

$$\phi = \tan^{-1} \left(\frac{y}{x} \right) = 108.43^\circ$$

$$A = (A_r, A_\theta, A_z) = \left[r \cos\phi \sin\theta + (r \cos\phi + z) \sin\theta \right] a_r \\ + \left[-r \sin\phi + (r \cos\phi + z) \cos\theta \right] a_\theta$$

$$r = \sqrt{40} ; \quad \tan\phi = -3, \quad \cos\phi = \frac{-2}{\sqrt{40}}, \quad \sin\phi = \frac{6}{\sqrt{40}}$$

$$A = \left\{ \sqrt{40} \cdot \frac{-2}{\sqrt{40}} \cdot \frac{6}{\sqrt{40}} + \left(\sqrt{40} \cdot \frac{-2}{\sqrt{40}} + 3 \right) \cdot \frac{6}{\sqrt{40}} \right\} a_r + \left\{ -\sqrt{40} \cdot \frac{36}{40} \right. \\ \left. + \left[\left(\sqrt{40} \cdot \frac{-2}{\sqrt{40}} + 3 \right) \cdot \frac{-2}{\sqrt{40}} \right] a_\theta \right\}$$

$$A = -0.94 a_r - 6.008 a_\theta$$

Prob ÷ Vector $A = 5a_x + 4a_y + 3a_z$ and $B = 2a_x + 3a_y + 4a_z$ are situated at a point (x, y, z) Find a) $A+B$ b) $A \cdot B$ c) Angle b/w A & B d) $A \times B$.

Prob ÷ Given that $P = 2a_x - a_y - 2a_z$, $Q = 4a_x + 3a_y + 2a_z$, $R = -a_x + a_y + 2a_z$ Find a) $|P+Q-R|$ b) $P \cdot Q \times R$ c) $Q \times P \cdot R$ d) $(P \times Q) \cdot (Q \times R)$ e) $(P \times Q) \times (Q \times R)$ f) $\cos\theta_{PR}$ g) $\sin\theta_{PQ}$.

The Inverse transformation $(A_r, A_\theta, A_\phi) \rightarrow (A_x, A_y, A_z)$ is easily obtained

$$\begin{bmatrix} A_x \\ A_y \\ A_z \end{bmatrix} = \begin{bmatrix} \sin\theta \cos\phi & \cos\theta \cos\phi & -\sin\phi \\ \sin\theta \sin\phi & \cos\theta \sin\phi & \cos\phi \\ \cos\theta & -\sin\theta & 0 \end{bmatrix} \begin{bmatrix} A_r \\ A_\theta \\ A_\phi \end{bmatrix}$$

$$\begin{bmatrix} A_r \\ A_\theta \\ A_\phi \end{bmatrix} = \begin{bmatrix} a_r \cdot a_x & a_r \cdot a_y & a_r \cdot a_z \\ a_\theta \cdot a_x & a_\theta \cdot a_y & a_\theta \cdot a_z \\ a_\phi \cdot a_x & a_\phi \cdot a_y & a_\phi \cdot a_z \end{bmatrix} \begin{bmatrix} A_x \\ A_y \\ A_z \end{bmatrix}$$

Problem: Given point $P(-2, 6, 3)$ and vector $A = y\mathbf{a}_x + (x+z)\mathbf{a}_y$ express A and P in cylindrical and spherical co-ordinates. evaluate P & A cylindrical & spherical systems.

Soln: $x = -2, y = 6, z = 3$

$$\rho = \sqrt{x^2 + y^2} = 6.32, \quad \phi = \tan^{-1}\left(\frac{y}{x}\right) = 108.43^\circ$$

$$z = 3.$$

$$r = \sqrt{x^2 + y^2 + z^2} = 7, \quad \theta = \tan^{-1}\left(\frac{\sqrt{x^2 + y^2}}{z}\right) = 64.62^\circ, \quad \phi = 108.43^\circ$$

$$P(-2, 6, 3) = P(6.32, 108.43^\circ, 3) = P(7, 64.6^\circ, 108.43^\circ)$$

In the Cartesian system A at P is

$$A = 6\mathbf{a}_x + \mathbf{a}_y$$

For Vector A , $A_x = y$; $A_y = x+z$; $A_z = 0$ Hence in cylindrical

$$\begin{bmatrix} A_r \\ A_\phi \\ A_z \end{bmatrix} = \begin{bmatrix} \cos\phi & \sin\phi & 0 \\ -\sin\phi & \cos\phi & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} y \\ x+z \\ 0 \end{bmatrix}$$

$$A_r = y \cos\phi + (x+z) \sin\phi$$

$$A_\phi = -y \sin\phi + (x+z) \cos\phi$$

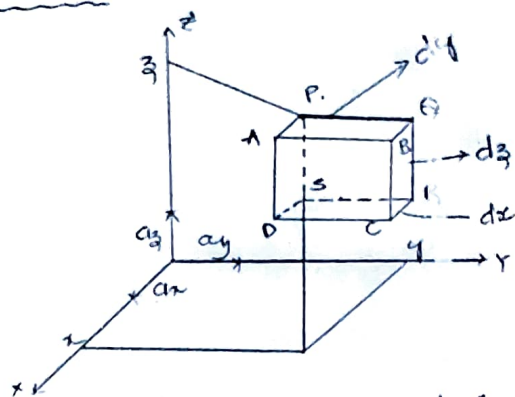
$$A_z = 0.$$

But $x = \rho \cos\phi$, $y = \rho \sin\phi$ substitute above eqn

Differential Length, Area, and Volume ÷

Differential elements in length, area and volume are useful in vector calculus. They are defined in the cartesian, cylindrical and spherical co-ordinate system.

Cartesian co-ordinates ÷



÷ Differential displacement ÷

If we move from point P to Q (or Q to P) $dL = dy a_y$. Because we are moving in the y-direction and if we move from S to Q. $dL = dy a_y + dz a_z$. Because we have to move dy along y, dz along z, and $dx = 0$ (no movement along x). Similarly, to move from D to Q it means (D to S $\rightarrow dx a_x$; S to P $\rightarrow dz a_z$; P to Q $\rightarrow dy a_y$) that

$$dL = dx a_x + dy a_y + dz a_z$$

∴ Differential displacement is given by

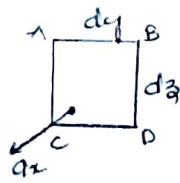
$$dL = dx a_x + dy a_y + dz a_z$$

Differential normal area ÷

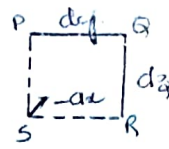
The Differential surface (or) area element ds may generally be defined as

$$ds = ds a_n$$

where ds is the area of the surface element and a_n is a unit-vector normal to the surface ds .



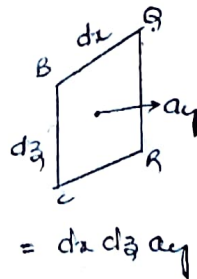
(or)



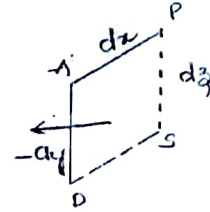
If we consider surface ABCD then $ds = dy dz ax$

where as for surface PQRS, $ds = -ax dy dz$ ($-ax$ is normal to PQRS).

Similarly

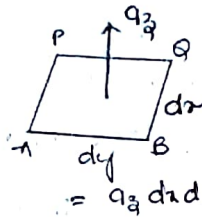


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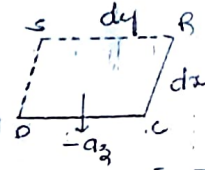


$$= dx dz ay$$

$$= -ay dx dz$$



(or)



$$= az dy dz$$

$$= -az dy dz$$

$\therefore$ Differential normal area is given by $ds = ax dy dz$

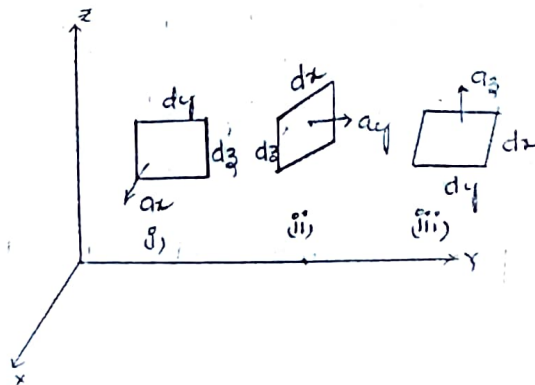
$$ay dx dz$$

$$az dx dy$$

Differential Volume

$$\text{volume (V)} = x y z$$

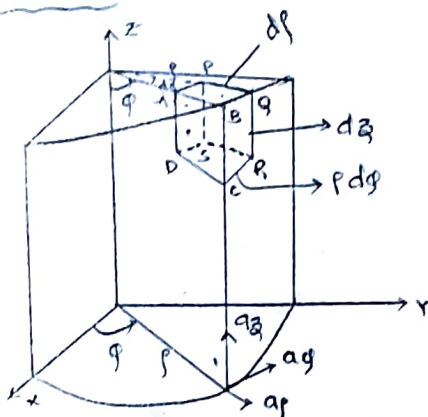
Differential volume is given by $dv = dx dy dz$



$\therefore$ Differential normal area in cartesian co-ordinates :-

$$i) ds = dy dz ax \quad ii) ds = dx dz ay \quad iii) ds = dx dy az$$

Cylindrical co-ordinates :



if we move from point P to Q or Q to P $dL = dr a_r$

Because we are moving in the r direction and if we move from A to Q, $dL = a_\phi r d\phi + dr a_r$. Because we have to move $d\phi$ along ϕ , dr along r and $dz = 0$

(no movement along z). Similarly, to move from D to Q it means (D to A $\rightarrow dz a_z$; A to D $\rightarrow r d\phi a_\phi$; D to Q $\rightarrow dr a_r$)

that

$$dL = dr a_r + r d\phi a_\phi + dz a_z$$

Differential displacement is given by

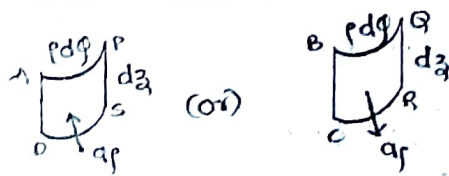
$$dL = dr a_r + r d\phi a_\phi + dz a_z$$

Differential normal area :

The differential surface or area element ds may generally be defined as

$$ds = ds a_n$$

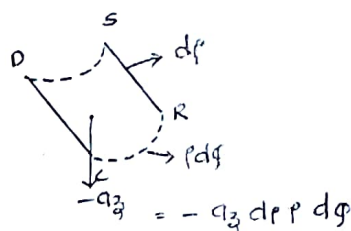
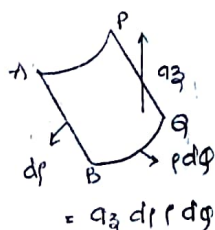
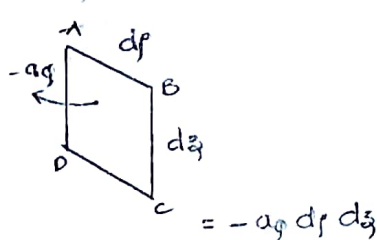
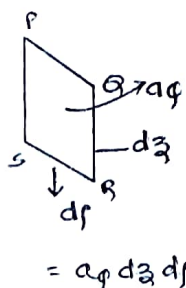
where ds is the area of surface element and a_n is a unit vector normal to the surface ds .



if we consider surface BGRC. then $ds = r dr dz \phi$

where as for surface ARSD $ds = -r dr dz \phi$

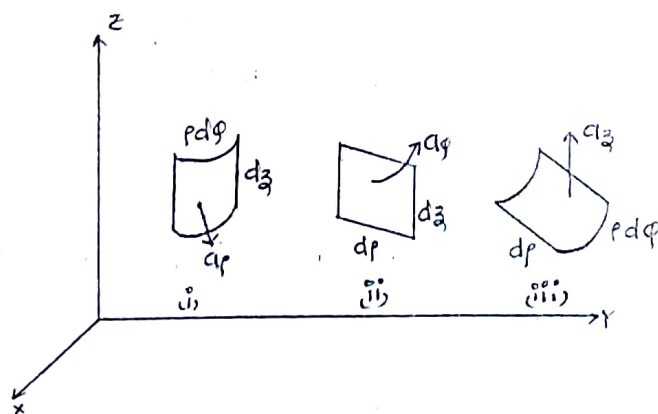
Similarly



$\therefore$ Differential normal area is given by $ds = r dr dz \phi$

$dr dz a_\phi$

$r dr dr a_z$



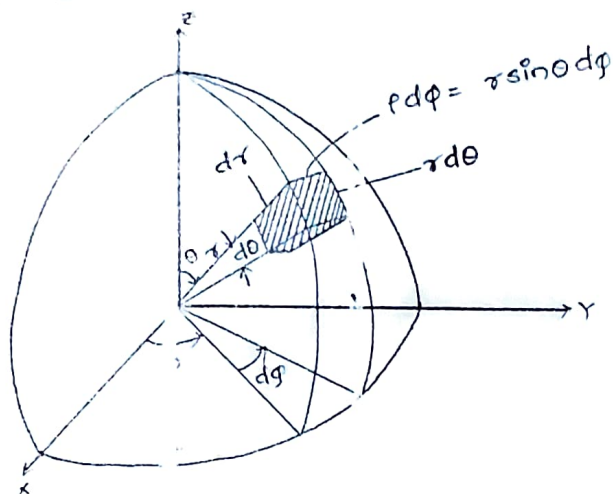
$\therefore$ Differential normal area in cylindrical co-ordinates $\div$

(i) $ds = r dr dz \phi$ (ii) $ds = a_\phi r dr dz$ (iii) $ds = a_z r dr d\phi$

$\therefore$ Differential volume $\div$

$$dv = r dr d\phi dz$$

spherical co-ordinates ÷

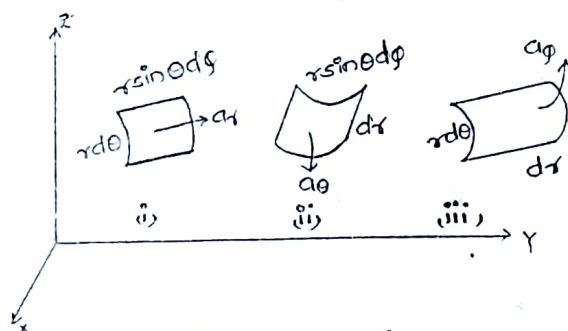


i) The differential displacement is given by

$$d\mathbf{l} = dr \mathbf{a}_r + r d\theta \mathbf{a}_\theta + r \sin\theta d\phi \mathbf{a}_\phi$$

ii) The differential surface or area is given by

$$ds = r^2 \sin\theta d\theta d\phi \mathbf{a}_r \\ r \sin\theta dr d\phi \mathbf{a}_\theta \\ r dr d\theta d\phi$$



Differential normal area in spherical co-ordinates

i) $ds = r^2 \sin\theta d\theta d\phi \mathbf{a}_r$ ii) $ds = r \sin\theta dr d\phi \mathbf{a}_\theta$

iii) $ds = r dr d\theta d\phi$

iii) The Differential volume is

$$dv = r^2 \sin\theta dr d\theta d\phi$$

DEL Operator :

The Del Operator, written ∇ , is the vector differential operator. In cartesian co-ordinates,

$$\nabla = \frac{\partial}{\partial x} a_x + \frac{\partial}{\partial y} a_y + \frac{\partial}{\partial z} a_z \quad \text{--- (1)}$$

This vector differential operator, otherwise known as the gradient operator, the operator is useful in defining

1. The gradient of a scalar V , written as ∇V
2. The divergence of a vector A , written as $\nabla \cdot A$
3. The curl of a vector A , written as $\nabla \times A$
4. The Laplacian of a scalar V , written as $\nabla^2 V$

$$a_x = \cos\phi a_r - \sin\phi a_\phi$$

$$a_y = \sin\phi a_r + \cos\phi a_\phi$$

$$a_z = a_z$$

$$\frac{\partial}{\partial x} = \cos\phi \frac{\partial}{\partial r} - \frac{\sin\phi}{r} \frac{\partial}{\partial \phi}$$

$$\frac{\partial}{\partial y} = \sin\phi \frac{\partial}{\partial r} + \frac{\cos\phi}{r} \frac{\partial}{\partial \phi}$$

$$\frac{\partial}{\partial z} = \frac{\partial}{\partial z}$$

$\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}$ sub in eqn no ①

$$\nabla = \left[\cos\phi \frac{\partial}{\partial r} - \frac{\sin\phi}{r} \frac{\partial}{\partial \phi} \right] a_x + \left[\sin\phi \frac{\partial}{\partial r} + \frac{\cos\phi}{r} \frac{\partial}{\partial \phi} \right] a_y + \frac{\partial}{\partial z} a_z$$

$$\nabla = \frac{\partial}{\partial r} [\cos\phi a_x + \sin\phi a_y] + \frac{\partial}{\partial \phi} \left[\frac{\cos\phi}{r} a_y - \frac{\sin\phi}{r} a_x \right] + \frac{\partial}{\partial z} a_z$$

$$\nabla = \frac{\partial}{\partial r} a_r + \frac{\partial}{\partial \phi} \frac{1}{r} a_\phi + \frac{\partial}{\partial z} a_z \quad \therefore \nabla \text{ in terms of } r, \phi, z$$

Wk, to obtain ∇ in terms of r, θ and ϕ

$$\frac{\partial}{\partial x} = \sin\theta \cos\phi \frac{\partial}{\partial r} + \frac{\cos\theta \cos\phi}{r} \frac{\partial}{\partial \theta} - \frac{\sin\phi}{r} \frac{\partial}{\partial \phi}$$

$$\frac{\partial}{\partial y} = \sin\theta \sin\phi \frac{\partial}{\partial r} + \frac{\cos\theta \sin\phi}{r} \frac{\partial}{\partial \theta} + \frac{\cos\phi}{r} \frac{\partial}{\partial \phi}$$

$$\frac{\partial}{\partial z} = \cos\theta \frac{\partial}{\partial r} - \frac{\sin\theta}{r} \frac{\partial}{\partial \theta}$$

UNIT-I

columns

Coulomb's Law states that the force F between two point charges Q_1 and Q_2 is

1. Along the line joining them
2. Directly proportional to the product $Q_1 Q_2$ of the charge
3. Inversely proportional to the square of the distance R b/w

Expressed mathematically,

$$F = \frac{k Q_1 Q_2}{R^2}$$

Where k is the proportionality constant.

In SI units, charges Q_1 and Q_2 are in coulombs (C),

the distance R in meters (m)

the force F in newtons (N) so that

$$k = \frac{1}{4\pi\epsilon_0}$$

the constant ϵ_0 is known as the permittivity of free space (in farads per meter) and has the value

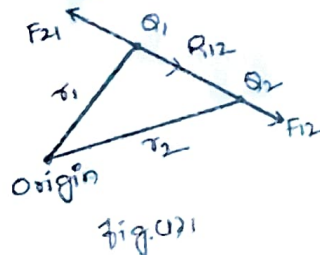
$$\epsilon_0 = 8.854 \times 10^{-12} \approx \frac{10^{-9}}{36\pi} \text{ F/m}$$

$$\text{or } k = \frac{1}{4\pi\epsilon_0} \approx 9 \times 10^9 \text{ N/F}$$

$$\therefore F = \frac{Q_1 Q_2}{4\pi\epsilon_0 R^2}$$

If point charges Q_1 and Q_2 are located at points having position vector r_1 and r_2 then the force F_{12} on Q_2 due to Q_1 shown in Fig(1), is given by

$$F_{12} = \frac{Q_1 Q_2}{4\pi\epsilon_0 R_{12}^2} \hat{R}_{12} \quad \text{--- (1)}$$



where

$$R_{12} = r_2 - r_1$$

$$R = |R_{12}|$$

$$a_{R12} = \frac{R_{12}}{R}$$

sub above equations in eqn no ① then

$$F_{12} = \frac{Q_1 Q_2}{4\pi\epsilon_0 R^3} R_{12}$$

(or)

$$F_{12} = \frac{Q_1 Q_2 (r_2 - r_1)}{4\pi\epsilon_0 |r_2 - r_1|^3}$$

→ Shown in fig(1), the force F_{21} on Q_1 due to Q_2 is given by

$$F_{21} = |F_{12}| a_{R12} = |F_{12}| (-a_{R12})$$

(or)

$$F_{21} = -F_{12}$$

Since

$$a_{R21} = -a_{R12}$$

If we have more than two point charges, with the help of principle of Superposition theorem to determine the force on a particular charge the principle states that if there are N charges $Q_1, Q_2, Q_3, \dots, Q_N$ located, respectively, at points with position vectors $r_1, r_2, \dots, r_N$ the resultant force F on a charge Q located at point r is the vector sum of the forces exerted on Q by each of the charges $Q_1, Q_2, \dots, Q_N$.

$$F = \frac{Q Q_1 (r - r_1)}{4\pi\epsilon_0 |r - r_1|^3} + \frac{Q Q_2 (r - r_2)}{4\pi\epsilon_0 |r - r_2|^3} + \dots + \frac{Q Q_N (r - r_N)}{4\pi\epsilon_0 |r - r_N|^3}$$

(or)

$$F = \frac{Q}{4\pi\epsilon_0} \sum_{k=1}^N \frac{Q_k (r - r_k)}{|r - r_k|^3}$$

Electric field Intensity $\div$ the electric field intensity or electric field strength E is the force per unit charge when placed in the electric field.

$$E = \frac{F}{Q}$$

The electric field intensity E is obviously in the direction of the force F and is measured in newtons / coulomb or volt/meter. The electric field intensity at point r due to a point charge located at r' is

$$E = \frac{Q}{4\pi\epsilon_0 R^2} \hat{a}_R = \frac{Q(r-r')}{4\pi\epsilon_0 |r-r'|^3} \quad \text{--- } \otimes \otimes$$

For N point charges $Q_1, Q_2, \dots, Q_N$ located at $r_1, r_2, \dots, r_N$, the electric field intensity at point r is obtained

$$E = \frac{Q_1(r-r_1)}{4\pi\epsilon_0 |r-r_1|^3} + \frac{Q_2(r-r_2)}{4\pi\epsilon_0 |r-r_2|^3} + \dots + \frac{Q_N(r-r_N)}{4\pi\epsilon_0 |r-r_N|^3}$$

$$(or) \quad E = \frac{1}{4\pi\epsilon_0} \sum_{k=1}^N \frac{Q_k(r-r_k)}{|r-r_k|^3}$$

Prob: Point charges 1mc and -2mc are located at $(3, 2, -1)$ and $(-1, -1, 4)$ respectively. Calculate the electric force on a 10-nC charge located at $(0, 3, 1)$ and the electric field intensity at that point.

soln: $F = \sum_{k=1,2} \frac{Q Q_k}{4\pi\epsilon_0 R^2} \hat{a}_R = \sum_{k=1,2} \frac{Q Q_k (r-r_k)}{4\pi\epsilon_0 |r-r_k|^3}$

$$F = \frac{Q}{4\pi\epsilon_0} \left\{ \frac{10^{-3} [(0, 3, 1) - (3, 2, -1)]}{|(0, 3, 1) - (3, 2, -1)|^3} - \frac{2 \cdot 10^{-3} [(0, 3, 1) - (-1, -1, 4)]}{|(0, 3, 1) - (-1, -1, 4)|^3} \right\}$$

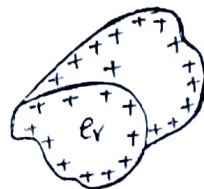
$$F = \frac{10^{-3} \times 10^{-9}}{4\pi \times 10^{-9}} \left[\frac{(-3, 1, 2)}{(9+1+4)^{3/2}} - \frac{2(1, 4, -3)}{(1+16+9)^{3/2}} \right]$$

$$F = -6.507 \hat{a}_x - 3.817 \hat{a}_y + 7.50 \hat{a}_z \text{ mN}$$

$$E = (-6.507, -3.817, 7.506) \cdot \frac{10^3}{10 \cdot 10^9}$$

Electric fields due to continuous charge distributions :-

various charge distributions and charge elements:



To denote the line charge density, surface charge density and volume charge density by ρ_L (in C/m), ρ_s (in C/m²) and ρ_v (in C/m³), respectively.

$$dq = e_L dl \rightarrow Q = \int_L e_L dl \quad (\text{Line charge})$$

$$dq = \epsilon_s ds \rightarrow Q = \int_S \epsilon_s ds \quad (\text{surface charge})$$

$$dq = \rho_r dv \rightarrow Q = \int_V \rho_r dv \quad (\text{volume charge})$$

The electric field intensity due to each of the charge distributions ρ_L , ρ_S , and ρ_V

the summation of the field contributed by the numerous¹⁵ point charges making up the charge distribution.

thus by replacing Q in eq. (1) with charge element $dQ = \rho_L dL$, $\rho_s ds$ or $\rho_v dv$ and integrating we get

$$E = \int \frac{\rho_L dL}{4\pi\epsilon_0 R^2} \mathbf{a}_R \quad \text{Line charge}$$

$$E = \int \frac{\rho_s ds}{4\pi\epsilon_0 R^2} \mathbf{a}_R \quad \text{Surface charge}$$

$$E = \int \frac{\rho_v dv}{4\pi\epsilon_0 R^2} \mathbf{a}_R \quad \text{Volume charge}$$

It should be noted that R^2 and $\mathbf{a}_R$ vary as the integral are evaluated.

Line charge :-

Consider a line charge with uniform charge density ρ_L extending from A to B along the z -axis as shown in fig. The charge element dQ associated with element $dL = dz$ of the line is

$$dQ = \rho_L dL = \rho_L dz$$

$$Q = \int \rho_L dz$$

$$Q = \int_{z_a}^{z_b} \rho_L dz$$

$$E = \int \frac{\rho_L dL}{4\pi\epsilon_0 R^2} \mathbf{a}_R$$

$$\mathbf{R} = (x, y, z) - (0, 0, z')$$

$$\mathbf{R} = x\mathbf{a}_x + y\mathbf{a}_y + (z - z')\mathbf{a}_z$$

$$|\mathbf{R}| = \sqrt{x^2 + y^2 + (z - z')^2}$$

$$R^2 = x^2 + y^2 + (z - z')^2$$

$$R^2 = r^2 + (z - z')^2$$

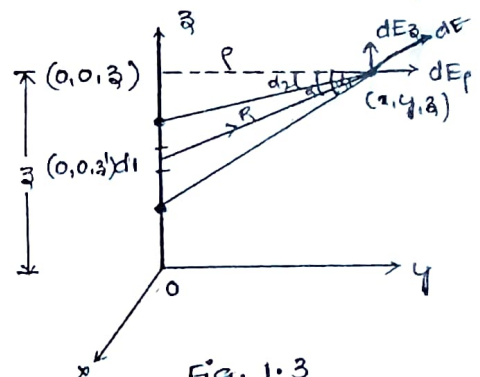


Fig. 1.3

$$\cos \alpha = \frac{p}{R}$$

$$R = p \sec \alpha$$

$$\tan \alpha = \frac{z - z'}{p}$$

$$z - z' = p \tan \alpha$$

$$\frac{q_R}{R^2} = \frac{R}{R^3} = \frac{[q_1 + (z-z')q_2]}{[r^2 + (z-z')^2]^{3/2}}$$

$$E = \int \frac{\rho_L dz'}{4\pi\epsilon_0} \frac{[q_1 + (z-z')q_2]}{[r^2 + (z-z')^2]^{3/2}}$$

$$E = \frac{\rho_L}{4\pi\epsilon_0} \int \frac{[q_1 + (z-z')q_2]}{[r^2 + (z-z')^2]^{3/2}} dz'$$

$$z = z - r \tan \alpha$$

$$\therefore z - z' = r \tan \alpha$$

$$dz' = 0 - r \sec^2 \alpha d\alpha$$

$$E = \frac{\rho_L}{4\pi\epsilon_0} \int \frac{[R \cos \alpha q_1 + R \sin \alpha q_2]}{(R^2)^{3/2}} (-) r \sec^2 \alpha d\alpha$$

$$\therefore r = R \cos \alpha$$

$$\therefore z - z' = r \tan \alpha$$

$$E = \frac{(-) \rho_L}{4\pi\epsilon_0} \int \frac{R [\cos \alpha q_1 + \sin \alpha q_2]}{R^3} r \sec^2 \alpha d\alpha$$

$$\frac{z - z'}{\tan \alpha} = r$$

$$\frac{z - z'}{\tan \alpha} = R \cos \alpha$$

$$\frac{z - z'}{\sin \alpha} \cdot \cos \alpha = R \cos \alpha$$

$$\therefore z - z' = R \sin \alpha$$

$$E = \frac{(-) \rho_L}{4\pi\epsilon_0} \int \frac{R^2 [\cos \alpha q_1 + \sin \alpha q_2]}{R^3} \sec^2 \alpha d\alpha$$

$$E = \frac{(-) \rho_L}{4\pi\epsilon_0} \int \frac{[\cos \alpha q_1 + \sin \alpha q_2]}{R} \sec^2 \alpha d\alpha$$

$$\therefore \frac{r}{\cos \alpha} = R$$

$$E = \frac{(-) \rho_L}{4\pi\epsilon_0} \int_{\alpha_1}^{\alpha_2} \frac{[\cos \alpha q_1 + \sin \alpha q_2]}{r} d\alpha$$

$$E = \frac{(-) \rho_L}{4\pi\epsilon_0 r} \left[\int_{\alpha_1}^{\alpha_2} \cos \alpha q_1 d\alpha + \int_{\alpha_1}^{\alpha_2} \sin \alpha q_2 d\alpha \right]$$

$$E = \frac{(-) \rho_L}{4\pi\epsilon_0 r} \left[q_1 \sin \alpha \Big|_{\alpha_1}^{\alpha_2} - q_2 \cos \alpha \Big|_{\alpha_1}^{\alpha_2} \right]$$

$$E = \frac{(-) \rho_L}{4\pi\epsilon_0 r} \left[(\sin \alpha_2 - \sin \alpha_1) q_1 - (\cos \alpha_2 - \cos \alpha_1) q_2 \right]$$

The above eqⁿ represents finite line charge of electric field strength.

Consider infinite line charge A $(0, 0, -\infty)$, B $(0, 0, \infty)$

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$$\alpha_1 = -\frac{\pi}{2}, \alpha_2 = \frac{\pi}{2}$$

$$\therefore E = \frac{\rho_L}{2\pi\epsilon_0} \frac{a_r}{r}$$

Surface charge :

Consider an infinite sheet of charge in the xy -plane with uniform charge density ρ_s , the charge associated with an elemental area ds is

$$dQ = \rho_s ds$$

and hence the total charge is

$$Q = \int \rho_s ds$$

$$E = \int \frac{dQ}{4\pi\epsilon_0 R^2} a_R$$

$$dE = \frac{dQ}{4\pi\epsilon_0 R^2} a_R$$

$$dE = \frac{\rho_s ds}{4\pi\epsilon_0 R^2} a_R$$

$$R = \rho a_r + h a_z$$

$$|R| = \sqrt{\rho^2 + h^2}$$

$$R^2 = \rho^2 + h^2$$

$$R^3 = (\rho^2 + h^2)^{3/2}$$

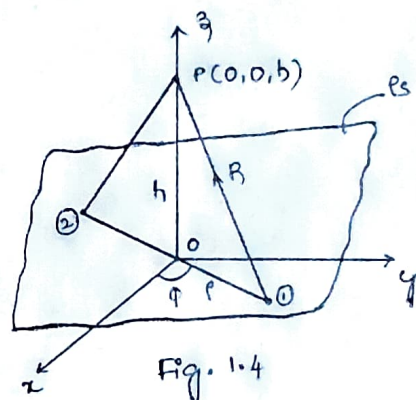
$$a_R = \frac{R}{|R|}$$

$$\frac{a_R}{R^2} = \frac{R}{R^3}$$

$$dE = \frac{\rho_s ds R}{4\pi\epsilon_0 R^3}$$

$$dE = \frac{\rho_s (\rho a_r + h a_z) ds}{4\pi\epsilon_0 (\rho^2 + h^2)^{3/2}}$$

$$dE = \frac{\rho_s h a_z ds}{4\pi\epsilon_0 (\rho^2 + h^2)^{3/2}}$$



$$dE = \frac{\rho_s}{4\pi\epsilon_0} \frac{h a_3}{(b^2 + r^2)^{3/2}} r d\phi dr a_3$$

$$\therefore ds = r d\phi dr a_3$$

$$E = \int \frac{\rho_s h a_3}{4\pi\epsilon_0 (b^2 + r^2)^{3/2}} r d\phi dr a_3$$

$$E = \frac{\rho_s}{4\pi\epsilon_0} \int_0^{2\pi} \int_0^\infty \frac{h a_3}{(b^2 + r^2)^{3/2}} r d\phi dr a_3$$

$$\therefore a_3 \cdot a_3 = 1$$

$$E = \frac{\rho_s}{4\pi\epsilon_0} \int_0^{2\pi} d\phi \int_0^\infty \frac{h r dr}{(b^2 + r^2)^{3/2}}$$

$$E = \frac{\rho_s}{4\pi\epsilon_0} (2\pi) \int_0^\infty \frac{h r dr}{(b^2 + r^2)^{3/2}}$$

$$E = \frac{\rho_s}{2\epsilon_0} \int_h^\infty \frac{h v dv}{(v^2)^{3/2}}$$

$$\text{let } r^2 + b^2 = v^2$$

$$2r dr = 2v dv$$

$$r=0 \rightarrow v=b$$

$$r=\infty \rightarrow v=\infty$$

$$E = \frac{\rho_s h}{2\epsilon_0} \left(-\frac{1}{v} \right) \Big|_b^\infty$$

$$\therefore \boxed{E = \frac{\rho_s}{2\epsilon_0} a_n}$$

where a_n is a unit vector normal to the sheet.

In a parallel plate capacitor, the electric field existing between the two plates having equal and opposite charges is given by

$$E = \frac{\rho_s}{2\epsilon_0} a_n + \frac{-\rho_s}{2\epsilon_0} (-a_n) = \frac{\rho_s}{\epsilon_0} a_n$$

Flux lines ÷ If a unit test charge is placed near a point charge, it experiences a force. The direction of this force can be represented by the lines, radially coming outward from a positive charge. These lines are called stream lines (or) flux lines.

Electric flux ÷ The total number of lines of force in any particular electric field is called the electric flux. It is represented by the symbol ψ . Similar to the charge, unit of electric flux is also Coulomb (C).

Properties of flux lines ÷

The electric flux is nothing but the lines of force, around a charge. Such electric flux lines have following properties,

1. The flux lines start from +ve charge and terminate on the -ve charge.

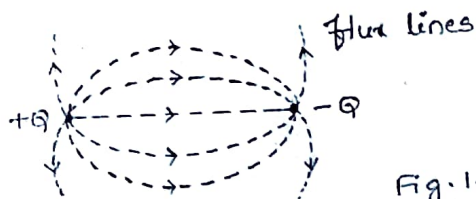


Fig. 1.5.1

2. If the -ve charge is absent, then the flux lines terminate at infinity.

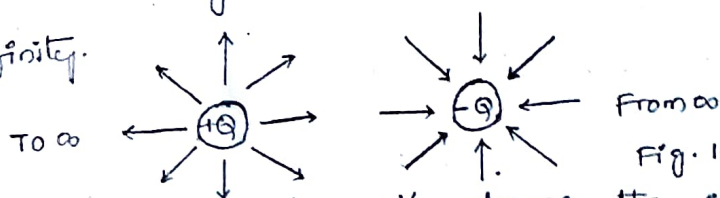


Fig. 1.5.2

while in absence of +ve charge, the electric flux terminates on the -ve charge from infinity.



3. These lines are parallel and never cross each other.
4. The lines are independent of the medium in which charges are placed.
5. Electric Flux (ψ) = Q coulombs

The flux is a scalar field. A vector field associated with the flux called electric flux density.

Electric Flux Density (E.D)

consider the two point charges. the flux lines originating from +ve charge and terminating at -ve charge. consider a unit surface area, the number of flux lines are passing through this surface area.

defn: The net flux passing normal through the unit surface area is called the electric flux density. It is denoted as $\vec{D}$. It has a specific direction which is normal to the surface area under consideration hence it is a vector field.

then electric flux density is defined as

$$D = \frac{\varphi}{s}$$

where $\varphi \rightarrow$ total flux

$s \rightarrow$ total surface area.

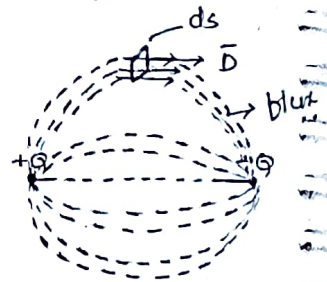


Fig. 1.5.3

As φ is measured in coulombs and s in square metres, the units of D are C/m^2 . this is also called displacement flux density (or) displacement density.

Vector Form of Electric Flux Density

consider the differential surface area ds at point P . the flux crossing through this differential area is $d\varphi$. the direction of $\vec{D}$ is same as that of direction of flux lines at that point. the differential area and flux lines are at right angles to each other at point P . Hence the direction of $\vec{D}$ is also normal to the surface area,

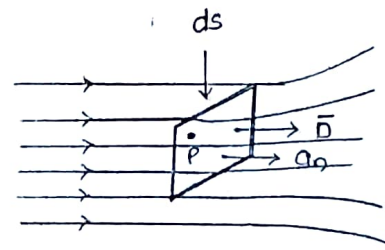


Fig. 1.5.4

in the direction of Unit Vector $\bar{a}_n$ which is normal to the surface area ds . Hence the flux density $\bar{D}$ at that point P can be represented in vector form as,

$$\bar{D} = \frac{dq}{ds} \bar{a}_n \text{ C/m}^2$$

where $dq \rightarrow$ total flux lines crossing normal through the differential area.

$ds \rightarrow$ Differential Surface area

$\bar{a}_n \rightarrow$ Unit Vector in the direction normal to the differential surface area.

Relationship between $\bar{D}$ and $\bar{E}$:

The electric field intensity $\bar{E}$ at a distance of r , from a point charge $+Q$ is given by,

$$\bar{E} = \frac{Q}{4\pi\epsilon_0 r^2} \bar{a}_r$$

Note: The electric flux density $\bar{D}$ at a distance of r , from a point charge $+Q$ is given by

$$\bar{D} = \frac{Q}{4\pi r^2} \bar{a}_r$$

Dividing the eqn of $\bar{D}$ and $\bar{E}$ due to point charge $+Q$

$$\frac{\bar{D}}{\bar{E}} = \frac{\frac{Q}{4\pi r^2} \bar{a}_r}{\frac{Q}{4\pi\epsilon_0 r^2} \bar{a}_r}$$

$$\bar{D} = \epsilon_0 \bar{E}$$

for free space

Generally $\bar{D} = \epsilon \bar{E}$

$$\epsilon = \epsilon_0 \epsilon_r$$

$$\bar{D} = \epsilon_0 \epsilon_r \bar{E}$$

Electric flux density for various charge distributions

$\vec{D}$ due to a point charge:

consider a point charge $+Q$ placed at the center of the imaginary sphere of radius r . The flux lines

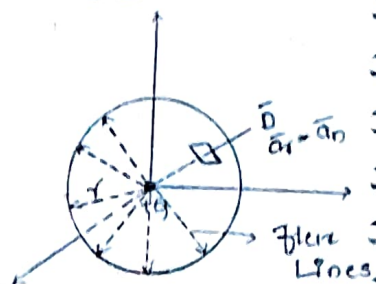


Fig. 1.6

originating from the point charge $+Q$ are directed radially outwards. The magnitude of the flux density at any point on the surface is,

$$\vec{D} = \frac{\text{Total flux } \psi}{\text{Total surface area } S}$$

But $\psi = Q = \text{total flux}$

$S = 4\pi r^2 = \text{total surface area}$

$$\vec{D} = \frac{Q}{4\pi r^2}$$

In vector form

$$\vec{D} = \frac{Q}{4\pi r^2} \vec{a}_r \text{ C/m}^2$$

$\vec{D}$ due to line charge:

consider a line charge having uniform charge density q_L C/m. Then the total charge along the line is given by

$$Q = \int_L q_L dL$$

$$\vec{D} = \frac{Q}{4\pi r^2} \vec{a}_r = \frac{\int_L q_L dL}{4\pi r^2} \vec{a}_r$$

if the line charge is infinite then $\vec{E}$ is

$$\vec{E} = \frac{q_L}{2\pi\epsilon_0 r} \vec{a}_r$$

$$D = \epsilon_0 \vec{E}$$

$$\vec{D} = \frac{q_L}{2\pi r} \vec{a}_r$$

Infinite line charge.

D due to Surface charge :-

consider a sheet of charge having uniform charge density of $\rho_s \text{ C/m}^2$. then the total charge on the surface is given by

$$Q = \int_S \rho_s \, ds$$

$$D = \frac{Q}{4\pi r^2} \bar{a}_r = \frac{\int_S \rho_s \, ds}{4\pi r^2} \bar{a}_r$$

The integration is over the surface S and is a double integral.

If the sheet of charge is infinite then $\bar{E}$ is derived as

$$\bar{E} = \frac{\rho_s}{2\epsilon_0} \bar{a}_n$$

$$D = \epsilon_0 E$$

$$D = \frac{\rho_s}{2} \bar{a}_n$$

Infinite sheet of charge.

D due to Volume charge :-

consider a charge enclosed by a volume, with a uniform charge density of $\rho_v \text{ C/m}^3$. then the total charge - enclosed by the volume is given by,

$$Q = \int_{vol} \rho_v \, dv$$

$$E = \frac{\int_{vol} \rho_v \, dv}{4\pi\epsilon_0 r^2} \bar{a}_r$$

$$D = \epsilon_0 E$$

$$D = \frac{\int_{vol} \rho_v \, dv}{4\pi r^2} \bar{a}_r$$

Gauss's Law

statement: The electric flux passing through any closed surface is equal to the total charge enclosed by that surface.

Mathematical Representation of Gauss's Law

The total charge enclosed by the irregular closed surface is Q coulombs. In any form of distribution. Hence the total flux that has to pass through the closed surface is Q .

consider a small differential surface ds

at point P . As the surface is irregular, the

direction of $\vec{D}$ as well as its magnitude is going to change from point to point on the surface. The surface ds under

consideration can be represented in the vector form in terms of its area and direction normal to the surface at that point.

$$\vec{ds} = ds \vec{a}_n$$

$\vec{a}_n$ = Normal to the surface ds at point P .

The flux $d\psi$ passing through the surface ds is the product of the component of $\vec{D}$ in the direction normal to the ds and $d\psi$.

Mathematically this can be represented as

$$d\psi = D_n ds$$

D_n → component of $\vec{D}$ in the direction of normal to the surface ds

$$D_n = |\vec{D}| \cos \theta$$

$$d\psi = |\vec{D}| \cos \theta ds$$

From the definition of the dot product

$$\vec{A} \cdot \vec{B} = |\vec{A}| |\vec{B}| \cos \theta_{AB}$$

$$\therefore |\vec{D}| ds \cos \theta = \vec{D} \cdot \vec{ds}$$

$$d\psi = \vec{D} \cdot \vec{ds}$$

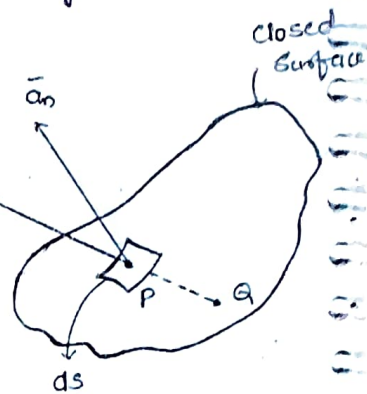


Fig. 1.7

This is the flux passing through incremental surface area ds .
Hence the total flux passing through the entire closed surface is to be obtained by finding the surface integration

$$\Psi = \int d\Psi = \oint \vec{D} \cdot d\vec{s}.$$

$\oint$ indicates the integration over the closed surface and called closed surface integral.

According to definition of Gauss Law $\Psi = \oint \vec{D} \cdot d\vec{s} = Q$.

The charge enclosed take any of the following forms.

1. If there are numbers of point charges $Q_1, Q_2, \dots, Q_N$ enclosed by the surface then

$$Q = Q_1 + Q_2 + Q_3 + \dots + Q_N = \sum Q_n$$

$$\Psi = Q = \sum Q_n$$

2. If there is a line charge with line charge density ρ_L then

$$\Psi = Q = \int \rho_L dL$$

3. If there is a surface charge with surface charge density ρ_s then,

$$\Psi = Q = \int \rho_s dS$$

4. If there is a volume charge with volume charge density ρ_v then

$$\Psi = Q = \int \rho_v dv$$

The common form used to represent Gauss's Law

$$\Psi = Q = \oint \vec{D} \cdot d\vec{s} = \int \rho_v dv$$

$$\oint \vec{D} \cdot d\vec{s} = \int \rho_v dv$$

↓
Apply Divergence theorem

$$\int \nabla \cdot \vec{D} dv = \int \rho_v dv$$

$\boxed{\nabla \cdot \vec{D} = \rho_v} \rightarrow$ is called as Maxwell eqⁿ.

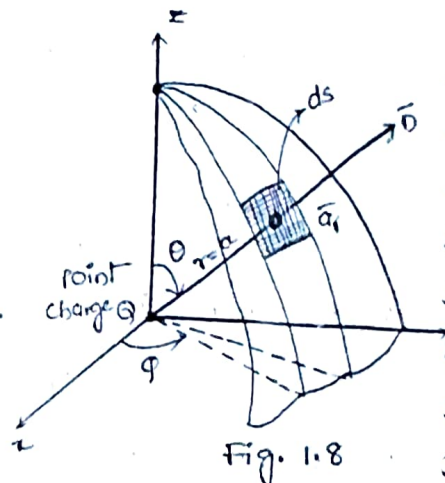
Applications of Gauss's Law.

point charge:

Let a point charge Q is located at the origin.

To determine $\vec{D}$ and to apply Gauss's Law, consider a spherical surface around Q , with centre as origin. This spherical surface is Gaussian surface and it satisfies required condition. The $\vec{D}$ is always directed radially outwards along $\vec{a}_r$ which is normal to the spherical surface at any point on the surface.

Consider a differential surface area ds the direction normal to the surface ds is $\vec{a}_r$, considering spherical co-ordinate-system. The radius of the sphere is $r=a$. The direction of $\vec{D}$ is along $\vec{a}_r$ which is normal to ds at any point P .



In spherical co-ordinate system, the ds normal to radial direction $\vec{a}_r$ is

$$ds = r^2 \sin\theta \, d\theta \, d\phi = a^2 \sin\theta \, d\theta \, d\phi \quad (\text{as } r=a)$$

$$ds = ds \vec{a}_r = a^2 \sin\theta \, d\theta \, d\phi \vec{a}_r$$

Now $\vec{D}$ due to the point charge is given by

$$\vec{D} = \frac{Q}{4\pi r^2} \vec{a}_r = \frac{Q}{4\pi a^2} \vec{a}_r \quad (\text{as } r=a)$$

$$\vec{D} \cdot d\vec{s} = |\vec{D}| |d\vec{s}| \cos\theta, \quad \theta_1 = 0$$

$$\vec{D} \cdot d\vec{s} = |\vec{D}| |d\vec{s}| \quad \because \theta_1 \text{ is the angle b/w } \vec{D} \text{ and } d\vec{s}$$

$$|\vec{D}| = \frac{Q}{4\pi a^2}, \quad |d\vec{s}| = a^2 \sin\theta \, d\theta \, d\phi$$

$$\vec{D} \cdot d\vec{s} = \frac{Q}{4\pi a^2} a^2 \sin\theta \, d\theta \, d\phi \cos 0$$

$$\vec{D} \cdot d\vec{s} = \frac{Q}{4\pi} \sin\theta \, d\theta \, d\phi$$

$$\vec{D} \cdot d\vec{s} = \frac{Q}{4\pi a^2} \vec{a}_r \cdot a^2 \sin\theta d\theta d\phi \vec{a}_r$$

$$= \frac{Q}{4\pi} \sin\theta d\theta d\phi [\vec{a}_r \cdot \vec{a}_r]$$

$$\text{But } \vec{a}_r \cdot \vec{a}_r = 1$$

$$\vec{D} \cdot d\vec{s} = \frac{Q}{4\pi} \sin\theta d\theta d\phi$$

$$\psi = \oint_S \vec{D} \cdot d\vec{s} = \int_0^{2\pi} \int_0^\pi \frac{Q}{4\pi} \sin\theta d\theta d\phi$$

$$\psi = \frac{Q}{4\pi} [-\cos\theta]_0^\pi [\phi]_0^{2\pi}$$

$$\psi = \frac{Q}{4\pi} [-(-1) - (-1)] 2\pi$$

$$\boxed{\psi = Q}$$

Gauss's Law is proved.

Use of Gauss's Law to obtain $\vec{D}$ and $\vec{E}$:

From Gauss's Law

$$Q = \oint_S \vec{D} \cdot d\vec{s}$$

The steps to obtain $\vec{D}$ and $\vec{E}$ are

1. Identify $|\vec{D}|$ and its direction
2. Identify $|d\vec{s}|$ and direction normal to $d\vec{s}$.
3. Take dot product $\vec{D} \cdot d\vec{s}$
4. choose the Gaussian Surface.
5. Integrate over the surface chosen as gaussian surface, keeping $|\vec{D}|$ unknown as it is.
6. find charge Q enclosed by Gaussian Surface.
7. Equate the charge Q , to the integration obtained with $|\vec{D}|$ as unknown.
8. Determine $|\vec{D}|$ and express $\vec{D}$ with its direction. Then $\vec{E} = \frac{\vec{D}}{\epsilon_0}$

For a sphere of radius r , the flux density $\vec{D}$ is in radial direction $\vec{a}_r$ always. Let $|\vec{D}| = D_r$

$$\vec{D} = D_r \vec{a}_r$$

Let the Gaussian Surface is sphere of radius r enclosing charge Q . While for the Gaussian surface i.e sphere of radius r , ds normal to $\vec{a}_r$ is

$$d\vec{s} = r^2 \sin\theta \, d\theta \, d\phi \, \vec{a}_r$$

$$\vec{D} \cdot d\vec{s} = D_r r^2 \sin\theta \, d\theta \, d\phi$$

$$\vec{a}_r \cdot \vec{a}_r = 1$$

Now integrate over the surface of sphere of constant radius

$$\oint_S \vec{D} \cdot d\vec{s} = \int_{\phi=0}^{2\pi} \int_{\theta=0}^{\pi} D_r r^2 \sin\theta \, d\theta \, d\phi$$

$$= D_r r^2 \left[-\cos\theta \right]_0^{\pi} \left[\phi \right]_0^{2\pi} = 4\pi r^2 D_r$$

$$\text{But } \oint_S \vec{D} \cdot d\vec{s} = Q$$

$$Q = 4\pi r^2 D_r$$

$$D_r = \frac{Q}{4\pi r^2} \text{ and hence}$$

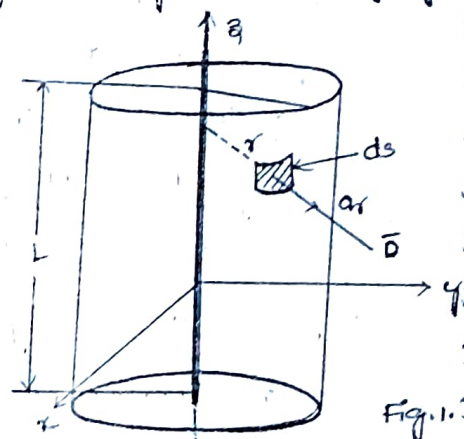
$$\vec{D} = D_r \vec{a}_r = \frac{Q}{4\pi r^2} \vec{a}_r$$

$$\boxed{E = \frac{D}{\epsilon_0} = \frac{Q}{4\pi\epsilon_0 r^2} \vec{a}_r}$$

Infinite line charge :-

consider an infinite line charge of density ρ_l cm lying along z -axis from $-\infty$ to $+\infty$.

consider the Gaussian surface as the right circular cylinder with z -axis as its axis and radius ρ . The flux density at any point on the surface is



directed radially outwards i.e. in the $\vec{a}_\rho$ direction according to cylindrical co-ordinate system. consider differential surface area ds , which is at a radial distance ρ from the line charge. The direction normal to ds is $\vec{a}_r$

As the line charge is along z-axis there can not be any component of $\vec{D}$ in z-direction. so $\vec{D}$ has only radial component.

$$\text{Now } Q = \oint \vec{D} \cdot d\vec{s}$$

The integration is to be evaluated for side surface, top surface and bottom surface.

$$Q = \oint_{\text{side}} \vec{D} \cdot d\vec{s} + \oint_{\text{top}} \vec{D} \cdot d\vec{s} + \oint_{\text{bottom}} \vec{D} \cdot d\vec{s}$$

$\vec{D} = D_r \hat{a}_r$ as has only radial component.

$d\vec{s} = \int d\phi dz \hat{a}_r$ normal to $\hat{a}_r$ direction.

$$\vec{D} \cdot d\vec{s} = D_r \int d\phi dz (\hat{a}_r \cdot \hat{a}_r) = D_r \int d\phi dz \quad \because \hat{a}_r \cdot \hat{a}_r = 1$$

Now D_r is constant over the side surface.

As $\vec{D}$ has only radial component and no component along $\hat{a}_z$ and $-\hat{a}_z$ hence integrations over top and bottom surface is zero.

$$\therefore \oint_{\text{top}} \vec{D} \cdot d\vec{s} = \oint_{\text{bottom}} \vec{D} \cdot d\vec{s} = 0.$$

$$\begin{aligned} Q &= \oint_{\text{side}} \vec{D} \cdot d\vec{s} = \oint_{\text{side}} D_r \int d\phi dz \\ &= \int_{z=0}^L \int_{\phi=0}^{2\pi} D_r \int d\phi dz = \int_0^L D_r [z]_0^L [\phi]_0^{2\pi} \end{aligned}$$

$$Q = 2\pi \int D_r L$$

$$D_r = \frac{Q}{2\pi L}$$

$$D = D_r \hat{a}_r = \frac{Q}{2\pi L} \hat{a}_r$$

$$\text{Put } \frac{Q}{L} = \rho_L \text{ C/m}$$

$$\vec{D} = \frac{\rho_L}{2\pi} \hat{a}_r \text{ C/m}^2$$

Due to infinite line charge

$$\boxed{E = \frac{D}{\epsilon_0} = \frac{\rho_L}{2\pi\epsilon_0} \hat{a}_r} \quad \text{V/m}$$

The results are same as obtained from the Coulomb's Law.

consider the two co-axial cylindrical conductors forming a co-axial cable. The radius of the inner conductor is 'a' while the radius of the outer conductor is 'b'. The length of cable is L.

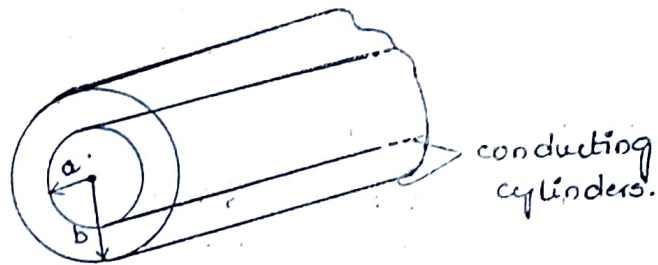


Fig. 1.10

The charge distribution on the outer surface of the inner conductor is having density ϵ_s C/m². The total outer-surface area of the inner conductor is $2\pi aL$.

Hence ϵ_s can be expressed in terms of ϵ_L .

$$\epsilon_L = \frac{\epsilon_s \times \text{Surface area}}{\text{Total length}} = \frac{\epsilon_s \times 2\pi aL}{L}$$

$$\epsilon_L = 2\pi a \epsilon_s \text{ C/m}$$

consider the right circular cylinder of length L as the Gaussian surface. Due to the symmetry, $\vec{D}$ has only radial component.

$$Q = D_r 2\pi r L \quad \text{--- (1)}$$

the total charge on the inner conductor is to be obtained by evaluating the surface integral of the surface charge distribution.

$$Q = \oint \epsilon_s ds$$

$$ds = r d\phi dz \quad \text{but } r = a$$

$$ds = a d\phi dz$$

$$Q = \int \epsilon_s a d\phi dz = \int_{z=0}^L \int_{\phi=0}^{2\pi} \epsilon_s a d\phi dz$$

$$Q = \epsilon_s a [z]_0^L [\phi]_0^{2\pi} = 2\pi aL \epsilon_s \quad \text{--- (2)}$$

Equating ① and ②

$$D_r 2\pi r L = 2\pi a L \rho_s$$

$$D_r = \frac{a \rho_s}{r}$$

It's act along radial direction i.e. $\vec{a}_r$

$$\vec{D} = \frac{a \rho_s}{r} \vec{a}_r$$

$$\text{But } \rho_s = \frac{Q}{2\pi a}$$

$$\vec{D} = \frac{a Q}{2\pi a r} \vec{a}_r = \frac{Q}{2\pi r} \vec{a}_r \text{ C/m}^2$$

$$\boxed{\vec{E} = \frac{Q}{2\pi \epsilon_0 r} \vec{a}_r} \quad \text{V/m}$$

Infinite sheet of charge:

consider the infinite sheet of charge of uniform charge density $\rho_s \text{ C/m}^2$, lying in the $z=0$ plane i.e. xy plane.

consider a rectangular box as a Gaussian surface which is cut by the sheet of charge to give $ds = dz dy$.

$\vec{D}$ acts normal to the plane i.e. $\vec{a}_n = \vec{a}_z$ and $-\vec{a}_n = -\vec{a}_z$ direction. Hence $D=0$ in x & y directions.

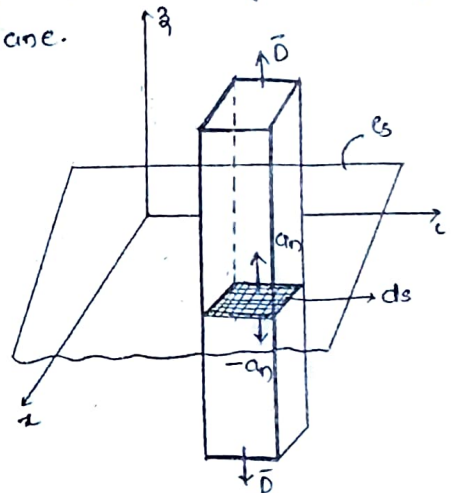


Fig. 1.11

Hence the charge enclosed can be written as

$$Q = \oint_S \vec{D} \cdot d\vec{s} = \oint_{\text{sides}} \vec{D} \cdot d\vec{s} + \oint_{\text{top}} \vec{D} \cdot d\vec{s} + \oint_{\text{bottom}} \vec{D} \cdot d\vec{s}$$

But $\oint_{\text{sides}} \vec{D} \cdot d\vec{s} = 0$ as $\vec{D}$ has no component in x & y direction

Now $\vec{D} = D_z \vec{a}_z$ for top surface

$$d\vec{s} = dz dy \vec{a}_z$$

$$\vec{D} \cdot d\vec{s} = D_z dz dy (\vec{a}_z \cdot \vec{a}_z) = D_z dz dy$$

$\vec{D} = D_z (-\vec{a}_z)$ for bottom surface

$$d\vec{s} = dz dy (-\vec{a}_z)$$

$$\vec{D} \cdot d\vec{s} = D_z dz dy (a_z \cdot a_z) = D_z dz dy$$

$$Q = \oint_{\text{top}} D_z dz dy + \oint_{\text{bottom}} D_z dz dy$$

$$\oint_{\text{top}} dz dy = \oint_{\text{bottom}} dz dy = 1 \rightarrow \text{Area of surface}$$

$$Q = 2D_z \cdot 1$$

$$Q = \rho_s \times 1 \quad \text{as } \rho_s \rightarrow \text{Surface charge density}$$

$$\rho_s = 2D_z$$

$$D_z = \frac{\rho_s}{2}$$

$$D = D_z a_z = \frac{\rho_s}{2} \bar{a}_z \text{ C/m}^2$$

$$E = \frac{D}{\epsilon_0} = \frac{\rho_s}{2\epsilon_0} \bar{a}_z \text{ V/m.}$$

Potential:-

Potential is defined as the work done per unit charge 24

$$V = \frac{W}{Q} \quad \frac{\text{Joules}}{\text{Coulombs}} \quad (\text{or}) \quad \text{Volts}$$

Electric Potential:-

From Coulomb's Law, the force on Q is $F = QE$. So that the work done in displacing the charge by dl is

$$W = \text{Force} \times \text{displacement}$$

$$dW = -F \cdot dl$$

$$\int dW = - \int F \cdot dl \Rightarrow - \int EQ \cdot dl \Rightarrow -Q \int E \cdot dl$$

$$\frac{W}{Q} = - \int E \cdot dl$$

$$V = - \int_A^B E \cdot dl$$

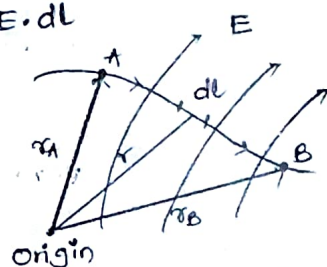


Fig. 1.12

Potential difference:-

The work done per unit charge in moving from one point to B in electric field is called as potential difference.

$$V_{AB} = - \int_A^B E \cdot dl$$

Potential due to point charge:-

E due to point charge (Q) located at origin then

$$E = \frac{Q}{4\pi\epsilon_0 r^2} \hat{r}$$

We know $V_{AB} = - \int_A^B E \cdot dl$

$$V_{AB} = - \int_A^B \frac{Q}{4\pi\epsilon_0 r^2} \hat{r} \cdot \hat{r} dr$$

$$V_{AB} = \frac{Q}{4\pi\epsilon_0} \int_{r_A}^{r_B} \frac{1}{r^2} dr$$

$$V_{AB} = \frac{Q}{4\pi\epsilon_0} \left[-\frac{1}{r_A} + \frac{1}{r_B} \right] \Rightarrow \frac{Q}{4\pi\epsilon_0} \left[\frac{1}{r_B} - \frac{1}{r_A} \right]$$

$$V_{AB} = V_B - V_A \quad \therefore V_B = \frac{Q}{4\pi\epsilon_0 r_B}, \quad V_A = \frac{Q}{4\pi\epsilon_0 r_A}$$

difference V_{AB} . If $V_A = 0$ as $r_A \rightarrow \infty$, the potential at any point ($r_B \rightarrow r$) due to a point charge Q located at the origin is

$$V = \frac{Q}{4\pi\epsilon_0 r}$$

Potential due to several point charges:-

For n point charges $Q_1, Q_2, \dots, Q_n$ located at point with position vectors $r_1, r_2, \dots, r_n$, the potential at r is

$$V(r) = \frac{Q_1}{4\pi\epsilon_0 |r-r_1|} + \frac{Q_2}{4\pi\epsilon_0 |r-r_2|} + \dots + \frac{Q_n}{4\pi\epsilon_0 |r-r_n|}$$

$$V(r) = \frac{1}{4\pi\epsilon_0} \sum_{k=1}^n \frac{Q_k}{|r-r_k|} \quad (\text{point charges})$$

Note:- The potential is not vector quantity. It is a scalar quantity. The electric scalar potential is V .

For continuous charge distributions replace Q_k with charge element q, dq, ds and dv

$$\therefore V(r) = \frac{1}{4\pi\epsilon_0} \int_L \frac{q_L(r') dl'}{|r-r'|} \quad (\text{Line charge})$$

$$V(r) = \frac{1}{4\pi\epsilon_0} \int_S \frac{q_S(r') ds'}{|r-r'|} \quad (\text{Surface charge})$$

$$V(r) = \frac{1}{4\pi\epsilon_0} \int_V \frac{q_V(r') dv'}{|r-r'|} \quad (\text{Volume charge})$$

Relation between E and V :-

From electric potential $(V) = - \int E \cdot dl$

$$dV = - E \cdot dl$$

$$dV = \frac{\partial V}{\partial x} dx + \frac{\partial V}{\partial y} dy + \frac{\partial V}{\partial z} dz$$

In cartesian co-ordinates,

$$E = E_x a_x + E_y a_y + E_z a_z$$

$$\therefore a_1 \cdot a_1 = 1$$

$$dL = dx a_x + dy a_y + dz a_z$$

$$dV = -E \cdot dL \Rightarrow -[E_x dx + E_y dy + E_z dz] \quad \text{--- (1)}$$

$$\text{But } dV = \frac{\partial V}{\partial x} dx + \frac{\partial V}{\partial y} dy + \frac{\partial V}{\partial z} dz \quad \text{--- (2)}$$

compare (1) and (2) equations

$$E_x = -\frac{\partial V}{\partial x}, \quad E_y = -\frac{\partial V}{\partial y}, \quad E_z = -\frac{\partial V}{\partial z}$$

$$\therefore E = -\frac{\partial V}{\partial x} a_x - \frac{\partial V}{\partial y} a_y - \frac{\partial V}{\partial z} a_z$$

$$E = -\left[\frac{\partial}{\partial x} a_x + \frac{\partial}{\partial y} a_y + \frac{\partial}{\partial z} a_z\right] V$$

$$\boxed{E = -\nabla V}$$

The potential V is scalar but the operator on V given in equation is vector and is called del operator - denoted as ∇ . The operation of del on a scalar V is called grad V .

$$\nabla = \frac{\partial}{\partial x} a_x + \frac{\partial}{\partial y} a_y + \frac{\partial}{\partial z} a_z$$

$$\boxed{E = -\nabla V}$$

Maxwell's Equations :-

From the Gauss's Law

$$Q = \oint D \cdot ds \quad \text{--- (1)}$$

while the charge enclosed in a volume is given by

$$Q = \int \rho_v dv \quad \text{--- (2)}$$

equating (1) & (2)

$$\oint D \cdot ds = \int \rho_v dv$$

↓
Apply divergence theorem

$\therefore \nabla \cdot D = \rho_v$ — is called Maxwell's eqⁿ.

From Potential The potential difference b/w points A and B is independent of the path taken. Hence

$$V_{BA} = -V_{AB}$$

$$\text{that is } V_{BA} + V_{AB} = \oint E \cdot dL = 0$$

$$\oint E \cdot dL = 0 \quad \text{--- (*)}$$

Fig.

This shows that the integral of E along a closed path. Physically, this implies that no net work is done in moving a charge along a closed path in an electrostatic field. Applying Stokes's theorem to eqⁿ (*)

$$\oint E \cdot dL = \int (\nabla \times E) \cdot dS = 0$$

$$\boxed{\nabla \times E = 0} \text{ — is called Maxwell's eqⁿ }$$

Linear Medium ÷

While the medium is called linear with respect to the electric field if the flux density $\vec{D}$ is directly proportional to the electric field intensity (E). The relationship is through the permittivity of the medium. If $\vec{D}$ is not directly proportional to E, the material is called non-linear.

Homogeneous Medium ÷

The medium is called homogeneous when the physical characteristics of the medium do not vary from point to point but remain same everywhere throughout the medium. If the characteristics vary from point to point the medium is called nonhomogeneous or heterogeneous.

Dielectric Constant $\div$ (ϵ_r) (Relative Permittivity)

It is the ratio of the permittivity of the dielectric to that of free space.

$$\epsilon_r = \frac{\epsilon}{\epsilon_0} \rightarrow \begin{array}{l} \text{permittivity of the dielectric} \\ \text{free space permittivity} \end{array}$$

Isotropic Material $\div$

Materials for which D and E are in the same direction are said to be isotropic. That is, isotropic dielectrics are those which have the same properties in all directions.

Dielectric Material $\div$ is linear if ϵ does not change with the applied E field, homogeneous if ϵ does not change from point to point, and isotropic if ϵ does not change with direction.

Energy Density $\div$ ($W_e = \frac{1}{2} \epsilon_0 E^2$)

When a test charge is moved from a point of lower potential to a point of higher potential in an electric field, the external agent moving the charge has to do work, thereby increasing the potential energy of the system, positive energy. The potential energy imparted to the system by external work on a charge when moved from lower potential to higher potential.

$$W = -Q \int E \cdot d\mathbf{l}$$

where W is the work done, E is electric field, $d\mathbf{l}$ is elemental distance and Q is test charge moved.

Let us consider three point charges Q_1 , Q_2 and Q_3 at infinity. In order to find the potential energy present in a system of these charges, must find the work done

Bring charge Q_1 from infinity to position 1, no work is done as the field does not exist in the system ($W_1=0$).

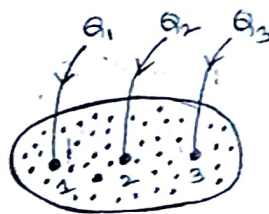


Fig 1.13

Bring Q_2 from infinity to position 2. This movement takes place in the field due to charge Q_1 . The amount of work is given by the product of Q_2 and the potential at that point due to Q_1 , work done to position $Q_2 = Q_2 V_{2,1}$, where $V_{2,1}$ is the potential at the location of Q_2 due to Q_1 , i.e. $W_2 = Q_2 V_{2,1}$.

To Bring charge Q_3 from infinity to position 3; movement is done in the field due to charges Q_1 and Q_2 , and the total work done to position Q_3 is $W_3 = Q_3 V_{3,1} + Q_3 V_{3,2}$; where $V_{3,1}$ and $V_{3,2}$ are the potentials at position 3 due to Q_1 and Q_2 respectively.

Total work done = potential energy of field

$$W_e = W_1 + W_2 + W_3$$

$$W_e = 0 + (Q_2 V_{2,1}) + (Q_3 V_{3,1} + Q_3 V_{3,2}) \quad \text{--- (1)}$$

If the three charges are brought to positions in the reverse order,

$$W_e = W_{e1} + W_{e2} + W_{e3}$$

$$W_e = 0 + (Q_2 V_{2,3}) + (Q_1 V_{1,3} + Q_1 V_{1,2}) \quad \text{--- (2)}$$

27
15

Adding (1) and (2) equations

$$W_e + W_e = (Q_1 V_{1,3} + Q_1 V_{1,2}) + (Q_2 V_{2,1} + Q_2 V_{2,3}) + (Q_3 V_{3,1} + Q_3 V_{3,2})$$

$V_{1,3} + V_{1,2}$ is the sum of potential at position Q_1 due to charges Q_3 and Q_2 respectively, i.e., total potential at $Q_1 = V_1$

$$\therefore 2W_e = Q_1 V_1 + Q_2 V_2 + Q_3 V_3$$

$$W_e = \frac{1}{2} (Q_1 V_1 + Q_2 V_2 + Q_3 V_3)$$

For a large number of charges, the total potential energy of a system as

$$W_e = \frac{1}{2} \sum_{i=1}^n Q_i V_i$$

For getting a general expression for the energy for the energy stored in a region of continuous charge distribution each charge is replaced by $\rho_v dv$, and the summation becomes an integral

$$W_e = \frac{1}{2} \int \rho_v V dv$$

Let a volume v containing a charge of volume charge density ρ_v is enclosed within a large sphere of radius

R.

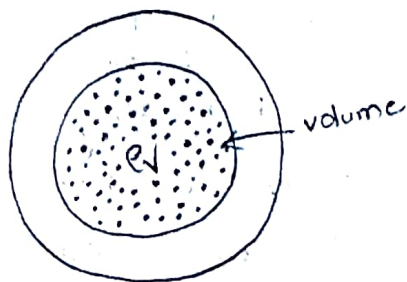


Fig 1.14

∴ volume charge distribution enclosed in a sphere ∴

$$W_e = \frac{1}{2} \int_{\text{spherical volume}} \epsilon_r v \, dv$$

$$W_e = \frac{1}{2} \int (\nabla \cdot D) v \, dv, \quad \because \nabla \cdot D = \rho_v \text{ by Gauss Law}$$

A general vector identity which is true for any scalar v and any vector function D is

$$\nabla \cdot (vD) = v(\nabla \cdot D) + D \cdot (\nabla v)$$

The above eqns can be written as

$$W_e = \frac{1}{2} \int_{\text{spherical volume}} (\nabla \cdot vD) \, dv - \frac{1}{2} \int_{\text{spherical volume}} (D \cdot \nabla v) \, dv$$

This holds for large radius R , and the plan is to let radius $R \rightarrow \infty$. By divergence theorem the first term

$$= \frac{1}{2} \int_{\text{sph vol}} (\nabla \cdot vD) \, dv = \frac{1}{2} \oint_{\text{sph vol}} vD \cdot ds$$

For a very large sphere, the volume charge looks like point charge. At very large R ; $D \propto \frac{1}{R^2}$, $v \propto \frac{1}{R}$ so $vD \propto \frac{1}{R^3}$

$$\frac{1}{2} \oint vD \cdot ds = 0 \quad R \rightarrow \infty$$

We have

$$W_e = -\frac{1}{2} \int_v (D \cdot \nabla v) \, dv$$

$$W_e = \frac{1}{2} \int_v (D \cdot E) \, dv \quad \text{where } \nabla v = -E$$

$$W_e = \frac{1}{2} \int_v \epsilon_0 E^2 \, dv$$

$$W_e = \frac{1}{2} \int_v \frac{D^2}{\epsilon_0} \, dv$$

It appears that each tiny volume dv be assigned the energy content $w_e \, dv$ where

$$w_e = \frac{1}{2} \epsilon_0 E^2$$

is called the energy

density for any field E . The units are joule/m^3 .

Current $\div$ the flow of charge per unit time i.e rate of flow of charge at a specified point or across a specified surface is called an electric current. It is measured in the unit ampere.

Mathematically

$$I = \frac{dq}{dt}$$

current density $\div$ (J) is a vector quantity, associated with the current and denoted as J .

The current density is defined as the current passing through the unit surface area, when the surface is held normal to the direction of the current. The current-density is measured in ampere per square metres (A/m^2)

$$\therefore I = \int_S \mathbf{J} \cdot d\mathbf{s}$$

Continuity Equation $\div$

The continuity equation of the current is based on the principle of conservation of charge. The principle states that the charges can neither be created nor be destroyed. Consider a closed surface S with a current density J , then the total current I crossing the surface S is

$$I = \oint_S \mathbf{J} \cdot d\mathbf{s}$$

The current flows outwards from the closed surface. It has been mentioned that the current means flow of positive charges. Hence the current I is constituted due to outward flow of +ve charges from the closed surface S . According to principle of conservation of charge

there must be decrease of an equal amount of the charge inside the closed surface. Hence the outward rate of flow of positive charge gets balanced by the rate of decrease of charge inside the closed surface.

$Q_i \rightarrow$ charge within the closed surface

$-\frac{dQ_i}{dt}$ — Rate of decrease of charge inside the closed Surface

Due to principle of conservation of charge, this rate of decrease is same as rate of outward flow of charge which is a current.

$$I = \oint_S \mathbf{J} \cdot d\mathbf{s} = -\frac{dQ_i}{dt}$$

This is the integral form of the continuity eqⁿ of the current. The -ve sign in above eqⁿ indicates outward flow of current from the closed surface.

If the current is entering the volume then

$$\oint_S \mathbf{J} \cdot d\mathbf{s} = -I = \frac{dQ_i}{dt}$$

Using the divergence theorem, convert the Surface-integral in integral form to the volume integral

$$\oint_S \mathbf{J} \cdot d\mathbf{s} = \int_{\text{Vol}} (\nabla \cdot \mathbf{J}) dv$$

$$-\frac{dQ_i}{dt} = \int_{\text{Vol}} (\nabla \cdot \mathbf{J}) dv$$

$$Q_i = \int_{\text{Vol}} \rho_v dv$$

where $\rho_v \rightarrow$ volume charge density

$$\therefore \int_{\text{Vol}} (\nabla \cdot \mathbf{J}) dv = -\frac{d}{dt} \left[\int_{\text{Vol}} \rho_v dv \right] = -\int_{\text{Vol}} \frac{\partial \rho_v}{\partial t} dv$$

for a constant surface, the derivative become the partial derivative.

$$\int_{\text{Vol}} (\nabla \cdot \mathbf{J}) dv = \int_{\text{Vol}} - \frac{\partial \rho_v}{\partial t} dv$$

$$\boxed{\nabla \cdot \mathbf{J} = - \frac{\partial \rho_v}{\partial t}}$$

this is the point form (or) differential form of the continuity equation of the current.

Relaxation Time :

consider a conducting material which is linear and homogeneous. The current density for such a material

$$\mathbf{J} = \sigma \mathbf{E} \quad \text{where } \sigma = \text{conductivity}$$

$$\mathbf{D} = \epsilon \mathbf{E}$$

$$\mathbf{E} = \frac{\mathbf{D}}{\epsilon}$$

$$\mathbf{J} = \sigma \frac{\mathbf{D}}{\epsilon} = \frac{\sigma}{\epsilon} \mathbf{D}$$

The point form of the continuity eqⁿ states that

$$\nabla \cdot \mathbf{J} = - \frac{\partial \rho_v}{\partial t}$$

$$\nabla \cdot \left(\frac{\sigma}{\epsilon} \mathbf{D} \right) = - \frac{\partial \rho_v}{\partial t}$$

$$\frac{\sigma}{\epsilon} \nabla \cdot \mathbf{D} = - \frac{\partial \rho_v}{\partial t}$$

$$\nabla \cdot \mathbf{D} = \rho_v$$

$$\frac{\sigma}{\epsilon} \rho_v = - \frac{\partial \rho_v}{\partial t}$$

$$\therefore \frac{\partial \rho_v}{\partial t} + \frac{\sigma}{\epsilon} \rho_v = 0$$

This is a differential eqⁿ ρ_v whose solution is given

$$\rho_v = \rho_0 e^{-(\sigma/\epsilon)t} = \rho_0 e^{-t/\tau}$$

where ρ_0 = charge density at $(t=0)$

electrons inside the given material, the charge density decays exponentially with a time constant $\tau = \frac{\epsilon}{\sigma}$ sec. This time is called relaxation time.

The relaxation time (τ) is defined as the time required by the charge density to decay to 36.8% of its initial value.

$$\tau = \text{Relaxation time} = \frac{\epsilon}{\sigma} \text{ sec.}$$

Poisson's and Laplace's Equations :-

The E and D in the given region are obtained using Coulomb's law and Gauss's Law. Using these laws is easy, if the charge distribution or potential throughout the region is known. Practically it is not possible in many situations, to know the charge distribution or potential variation throughout the region. Practically charge and potential may be known at some boundaries of the region, only. From those values it is necessary to obtain potential and E throughout the region. Such electrostatic problems are called boundary value problems. To solve such problems, Poisson's and Laplace's equations must be known.

From the Gauss's Law in the point form, Poisson's equation can be derived. Consider the Gauss's Law

$$\nabla \cdot D = \rho_v$$

where $D \rightarrow$ Flux density and $\rho_v \rightarrow$ volume charge density.

It is known that for a homogeneous, isotropic and linear medium, flux density and electric field intensity are directly proportional. Thus

$$D = \epsilon E$$

$$\nabla \cdot D = \rho_v$$

From the gradient relationship $E = -\nabla V$

$$\therefore \nabla \cdot E(-\nabla V) = \rho_v$$

$$-E[\nabla \cdot \nabla V] = \rho_v$$

$$\nabla \cdot \nabla V = -\frac{\rho_v}{\epsilon}$$

Now $\nabla \cdot \nabla$ operation is called del squared operation and denoted as ∇^2

$$\boxed{\nabla^2 V = -\frac{\rho_v}{\epsilon}} \rightarrow \text{is called Poisson's eq}$$

If in a certain region, volume charge density is zero ($\rho_v = 0$), which is true for dielectric medium then the Poisson's equation takes the form

$$\boxed{\nabla^2 V = 0} \text{ (For charge free region)}$$

This is special case of Poisson's equation and is called Laplace's equation. The ∇^2 operation is called the Laplacian of V .

Concept of Capacitance :-

Consider two conducting materials M_1 and M_2 which are placed in a dielectric medium having permittivity ϵ . Material M_1 carries a +ve charge Q while the material M_2 carries a -ve charge, equal in magnitude as Q . There are no other charges present and total charge on the system is zero. In conductors, charge can not

reside within the conductor and it resides only on the surface. Thus for M_1 and M_2 , charges $+Q$ and $-Q$ reside on the surfaces of M_1 and M_2 respectively.

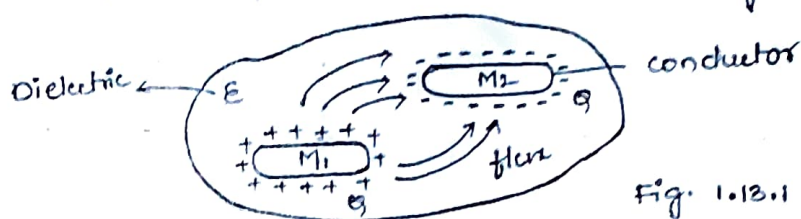


Fig. 1.12.1

Such a system which has two conducting surfaces - carrying equal and opposite charges, separated by a dielectric is called capacitive system giving rise to a capacitance.

The electric field is normal to the conductor surface and the electric flux is directed from M_1 towards M_2 in such a system. There exists a potential difference between the two surfaces of M_1 and M_2 . Let this potential is V_{12} . The ratio of the magnitudes of the total charge on any one of the two conductors and potential difference b/w the conductors is called the capacitance of the two conductor system denoted as C .

$$\text{In general } C = \frac{Q}{V}.$$

where $Q \rightarrow$ charge in coulombs

$V \rightarrow$ potential difference in volts.

The capacitance is measured in Farads (F) and

$$1 \text{ farad} = \frac{1 \text{ coulomb}}{1 \text{ volt.}}$$

As charge Q resides only on the surface of the conductor, it can be obtained from the Gauss's law

$$Q = \oint \mathbf{D} \cdot d\mathbf{s} = \oint \epsilon_0 \epsilon_s \mathbf{E} \cdot d\mathbf{s} = \oint \epsilon \mathbf{E} \cdot d\mathbf{s}$$

while V is the work done in moving unit $+ve$ charge from $-ve$ to the $+ve$ surface and can be obtained

$$V = - \int_L \mathbf{E} \cdot d\mathbf{L} = - \int_-^+ \mathbf{E} \cdot d\mathbf{L}$$

Hence Capacitance can be expressed as

$$C = \frac{Q}{V} = \frac{\oint \epsilon \mathbf{E} \cdot d\mathbf{s}}{\int_-^+ \mathbf{E} \cdot d\mathbf{L}}$$

Applications of Capacitance :-

Parallel plate capacitor :-

A parallel plate capacitor consists of two parallel metallic plates separated by distance 'd'. The space between the plates is filled with a dielectric

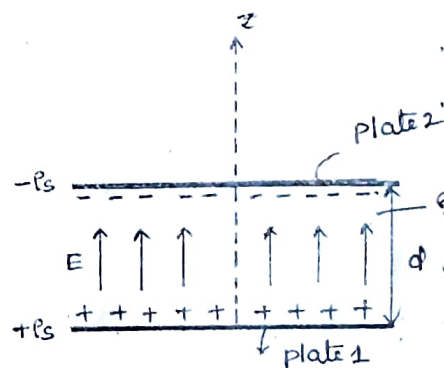


Fig. 1.13.2

of permittivity ϵ . The lower plate 1 carries the $+ve$ charge and is distributed over it with a charge density $+\rho_s$. The upper plate, i.e. plate 2 carries the $-ve$ charge and is distributed over its surface with a charge density $-\rho_s$. The plate 1 is placed in $z=0$ i.e. xy plane hence normal to its z -direction. While upper plate 2 is in $z=d$ plane parallel to xy plane.

Let $A \rightarrow$ Area of cross section of the plates

$$Q = \rho_s A$$

This is magnitude of charge on any one plate as charge carried by both is equal in magnitude. To find potential difference, let us obtain $\vec{E}$ between the plates.

Assuming plate 1 to be infinite sheet of charge

$$\vec{E}_1 = \frac{\rho_s}{2\epsilon} \vec{a}_z$$

As $\vec{E}$ is normal at the boundary b/w conductor and dielectric without any tangential component.

while for plate 2,

$$\vec{E}_2 = -\frac{\rho_s}{2\epsilon} (-\vec{a}_z)$$

The direction of $\vec{E}_2$ is downwards i.e. in $-\vec{a}_z$ direction
In between the plates

$$\vec{E} = \vec{E}_1 + \vec{E}_2 = \frac{\rho_s}{2\epsilon} \vec{a}_z + \frac{\rho_s}{2\epsilon} \vec{a}_z = \frac{\rho_s}{\epsilon} \vec{a}_z$$

The potential difference is given by

$$V = - \int_{\text{upper}}^{\text{lower}} \vec{E} \cdot d\vec{L} = - \int_{\text{upper}}^{\text{lower}} \frac{\rho_s}{\epsilon} \vec{a}_z \cdot d\vec{L}$$

Now $d\vec{L} = dz\vec{a}_z + dy\vec{a}_y + dx\vec{a}_x$ in Cartesian system

$$V = - \int_{z=d}^{z=0} \frac{\rho_s}{\epsilon} \vec{a}_z \cdot [dz\vec{a}_z + dy\vec{a}_y + dx\vec{a}_x]$$

$$V = - \int_{z=d}^{z=0} \frac{\rho_s}{\epsilon} dz = -\frac{\rho_s}{\epsilon} [z]_d^0$$

$$V = \frac{\rho_s d}{\epsilon}$$

$$\therefore C = \frac{Q}{V} = \frac{\rho_s A}{\frac{\rho_s d}{\epsilon}} = \frac{\epsilon A}{d}$$

thus if $\epsilon = \epsilon_0 \epsilon_r$

$$C = \frac{\epsilon_0 \epsilon_r A}{d}$$

$$V = - \frac{Q}{4\pi\epsilon} \left[\frac{1}{r} \right]_{r=b}^{r=a}$$

$$V = \frac{Q}{4\pi\epsilon} \left[\frac{1}{r} \right]_{r=b}^{r=a}$$

$$V = \frac{Q}{4\pi\epsilon} \left[\frac{1}{a} - \frac{1}{b} \right]$$

$$C = \frac{Q}{V}$$

$$\therefore C = \frac{4\pi\epsilon}{\frac{1}{a} - \frac{1}{b}}$$

Capacitance of single isolated sphere :-

consider a single isolated sphere of radius 'a' given a charge +Q. It forms a capacitance with an outer plate which is infinitely large hence $b = \infty$.

The capacitance of such a single isolated - spherical conductor can be obtained by substituting b

$$C = \frac{4\pi\epsilon}{\frac{1}{a} - \frac{1}{\infty}}$$

$$\text{but } \frac{1}{\infty} = 0$$

$$C = 4\pi\epsilon a$$

Coulomb's Law ÷

$$F \propto \frac{Q_1 Q_2}{d^2}$$

$$F = \frac{k Q_1 Q_2}{d^2}$$

$$k \rightarrow \text{proportionality Const.} = \frac{1}{4\pi\epsilon_0}$$

$$\epsilon_0 = 8.854 \times 10^{-12} \text{ F/m (free space Permittivity)}$$

$$\text{Vector form: } F_{12} = \frac{Q_1 Q_2}{4\pi\epsilon_0 R_{12}^2} \hat{a}_{12}$$

$$R_{12} = r_2 - r_1, \quad \hat{a}_{12} = \frac{R_{12}}{|R_{12}|}$$

$$F_{12} = \frac{Q_1 Q_2 (r_2 - r_1)}{4\pi\epsilon_0 |r_2 - r_1|^3}$$

If there are n charges Q_1 to Q_n then the resultant force on the charge Q is

$$F = \sum_{k=1}^n \frac{Q Q_k}{4\pi\epsilon_0 R_k^2} \hat{a}_{kq}$$

Electric Field Intensity ÷ (E)

$$E = \frac{F}{Q} \text{ i.e. Force per unit charge}$$

$$\text{Vector form: } E = \frac{Q}{4\pi\epsilon_0 R_{12}^2} \hat{a}_{12}$$

E is measured in N/C (or) V/m

If there are n charges Q_1 to Q_n then the resultant

E at a point is

$$E = \frac{1}{4\pi\epsilon_0} \sum_{k=1}^n \frac{Q_k}{R_k^2} \hat{a}_{kq}$$

Types of charge Distributions ÷

1. point charge
2. Line charge
3. Surface charge
4. volume charge

- It can be seen that the value of Capacitance depends
- The permittivity of the dielectric used.
 - The area of cross section of the plates.
 - The distance of separation of the plates.

Capacitance of a Co-axial cable :-

consider a co-axial cable (or) co-axial capacitor.

Let a = Inner radius

b = Outer radius

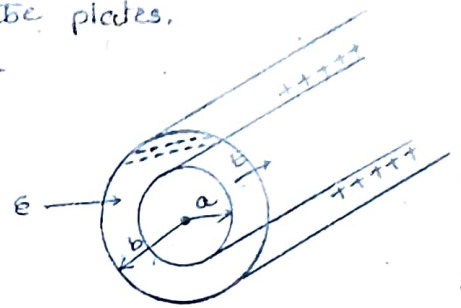


Fig. 1.13.3

The two concentric conductors are separated by dielectric of permittivity ϵ . The length of the cable is L . The inner conductor carries a charge density $+ \rho_L$ on its surface. The equal and opposite charge density $- \rho_L$ exists on the outer conductor.

$$Q = \rho_L L$$

Assuming cylindrical co-ordinate system, E will be radial from inner to outer conductor, and for infinite line charge it is given by

$$E = \frac{\rho_L}{2\pi\epsilon r}$$

E is directed from inner conductor to the outer conductor. The potential difference is work done in moving unit charge against E i.e. from $r=b$ to $r=a$.

To find potential difference, consider dL in radial direction which is dr or

$$V = - \int E \cdot dL = - \int_{r=b}^{r=a} \frac{\rho_L}{2\pi\epsilon r} dr$$

$$V = - \frac{\rho_L}{2\pi\epsilon} \left[\ln r \right]_b^a = - \frac{\rho_L}{2\pi\epsilon} \ln \left[\frac{a}{b} \right]$$

$$V = \frac{QL}{2\pi\epsilon} \ln \left[\frac{b}{a} \right]$$

$$C = \frac{Q}{V} = \frac{QL}{\frac{QL}{2\pi\epsilon} \ln \left[\frac{b}{a} \right]}$$

$$C = \frac{2\pi\epsilon L}{\ln \left[\frac{b}{a} \right]}$$

Spherical Capacitor :-

Consider a spherical capacitor formed two concentric spherical conducting shells of radius a and b . The radius

of outer sphere is ' b ' while that of

inner sphere is ' a '. Thus $b > a$. The region b/w the two spheres is filled with a dielectric of permittivity ϵ .

The inner sphere is given a positive charge $+Q$ while for the outer sphere it is $-Q$.

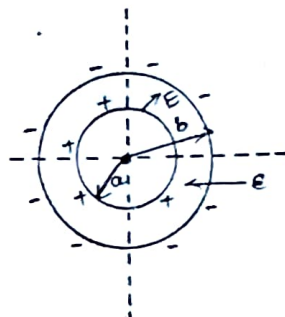


Fig. 1.13.4

Considering Gaussian surface as a sphere of radius r , it can be obtained that $\vec{E}$ is in radial direction and given by

$$E = \frac{Q}{4\pi\epsilon r^2} \text{ or}$$

The potential difference is work done in moving unit +ve charge against the direction of $\vec{E}$ i.e. from $r=b$ to $r=a$

$$V = - \int_{-}^{+} \vec{E} \cdot d\vec{l} = - \int_{r=b}^{r=a} \frac{Q}{4\pi\epsilon r^2} \text{ or} \cdot d\vec{l}$$

Now $d\vec{l} = dr \hat{r}$ → in radial direction.

$$V = - \int_{r=b}^{r=a} \frac{Q}{4\pi\epsilon r^2} \text{ or} \cdot dr$$

$$V = - \int_{r=b}^{r=a} \frac{Q}{4\pi\epsilon r^2} dr$$

Line charge $\vec{E}_L \rightarrow$ Line charge density in C/m 34

$$Q = \int_L \vec{E}_L dL = \text{Line integral}$$

Surface charge $\vec{E}_s \rightarrow$ surface charge density in C/m^2

$$Q = \int_s \vec{E}_s ds = \text{Surface integral}$$

this is double integral.

Volume charge $\vec{E}_v \rightarrow$ volume charge density in C/m^3

$$Q = \int_v \vec{E}_v dv \rightarrow \text{volume integral}$$

this is triple integral.

$\vec{E}$ due to line charge $\div$

$$dE = \frac{dQ}{4\pi\epsilon_0 R^2} q_R = \frac{\vec{E}_L dL}{4\pi\epsilon_0 R^2} q_R$$

$$\therefore E = \int_L \frac{\vec{E}_L dL}{4\pi\epsilon_0 R^2} q_R$$

for infinite line charge

$$E = \frac{\vec{E}_L}{4\pi\epsilon_0 r} q_R \quad V/m$$

$\vec{E}$ due to surface charge $\div$

$$dE = \frac{dQ}{4\pi\epsilon_0 R^2} q_R \rightarrow \frac{\vec{E}_s ds}{4\pi\epsilon_0 R^2} q_R$$

$$E = \int_s \frac{\vec{E}_s ds}{4\pi\epsilon_0 R^2} q_R$$

For infinite sheet of charge

$$E = \frac{\vec{E}_s}{2\epsilon_0} \bar{q}_R \quad V/m$$

$\vec{E}$ due to volume charge $\div$

$$dE = \frac{dQ}{4\pi\epsilon_0 R^2} q_R \rightarrow \frac{\vec{E}_v dv}{4\pi\epsilon_0 R^2} q_R$$

$$E = \int \frac{\vec{E}_v dv}{4\pi\epsilon_0 R^2} q_R$$

E and D for various

using

charge distribution

Field Intensity
E in V/m

Flux density D
in C/m²

$$\frac{Q}{4\pi\epsilon_0 r^2} a_r$$

$$\frac{Q}{4\pi r^2} a_r$$

Infinite line charge
having density ϵ_L C/m

$$\frac{\epsilon_L}{4\pi\epsilon_0 r} a_r$$

$$\frac{\epsilon_L}{2\pi r} a_r$$

Infinite surface charge
having density ϵ_s C/m²

$$\frac{\epsilon_s}{2\epsilon_0} a_n$$

$$\frac{\epsilon_s}{2} a_n$$

volume charge having
density ϵ_v C/m³

$$\frac{\int \epsilon_v dv}{4\pi\epsilon_0 r^2} a_r$$

$$\frac{\int \epsilon_v dv}{4\pi r^2} a_r$$

Gauss's Law :-

The electric flux passing through any closed surface is equal to the total charge enclosed by that surface.

Mathematically $\psi = Q = \oint_S D \cdot ds$

charge Configuration

point charge Q_1, Q_2, Q_3

Line charge ϵ_L

Surface charge ϵ_s

Volume charge ϵ_v

charge enclosed

$$\psi = Q = \sum_{n=1}^N Q_n$$

$$\psi = Q = \int_L \epsilon_L dL$$

$$\psi = Q = \int_S \epsilon_s ds$$

$$\psi = Q = \int_V \epsilon_v dv$$

Relation b/w D and E

$$D = \epsilon E$$

$\epsilon \rightarrow$ permittivity

$$\epsilon = \epsilon_0 \epsilon_r$$

$\epsilon_0 \rightarrow$ free space permittivity = 8.854×10^{-12} F/m

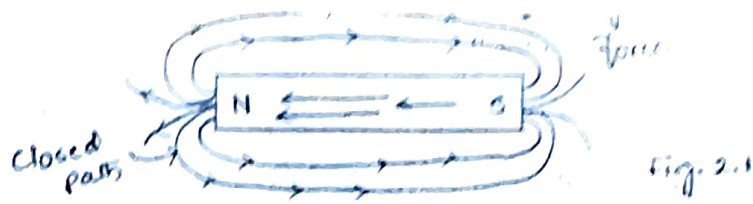
$\epsilon_r \rightarrow$ relative permittivity

$$D = \epsilon_0 E$$

if $\epsilon_r = 1$.

The electrostatic field exists due to the static charges i.e. charges at rest. The magnetic field exists due to a permanent magnet, which is a natural magnet. When the charges are in motion, they are surrounded by a magnetic field. The charges in motion i.e. flow of charges constitute an electric current. Thus a current carrying conductor is always surrounded by a magnetic field. If such a current flow is steady i.e. time invariant then the magnetic field produced is a steady magnetic field which is also a time invariant. The direct current (d.c) is a steady flow of current hence magnetic field produced by a conductor carrying a d.c current is a static steady magnetic field. The study of steady magnetic field, existing in a given space, produced due to the flow of direct current through a conductor is called magnetostatics.

Consider a permanent magnet. It has two poles, north (N) and south (S). The region around a magnet within which the influence of the magnet can be experienced is called magnetic field. The existence of such a field can be experienced with the help of compass. Such a field is represented by imaginary lines around the magnet which are called magnetic lines of force.



These lines of force are also called magnetic lines of flux or magnetic flux lines. Every magnetic flux line starting from north pole must end at south pole and complete the path from south to north internal to the magnet.

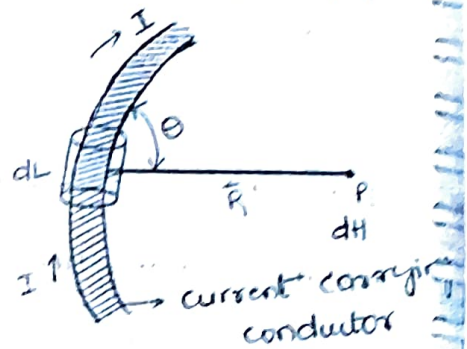
Magnetic Field Intensity $\div H$

Magnetic Field Intensity at any point in the magnetic field is defined as the force experienced by a unit north pole of one weber strength, when placed at that point. The magnetic flux lines are measured in webers while magnetic field intensity is measured in (N/Wb) or A/m . It is denoted as H . It is a vector quantity.

Biot Savart Law $\div$

The Biot-Savart Law allows us to obtain the differential magnetic field intensity dH , produced at a point P , due to differential current element IdL .

consider a differential length dL hence the differential current element is IdL . This is very small part of the current carrying conductor. The



point P is at a distance R from the differential current element. The θ is the angle b/w the differential current element and the line joining point P to the differential current

Use Biot-Savart Law states that,

The magnetic field intensity dH produced at a point P due to a differential current element IdL is

→ proportional to the product of the current I and differential length dL .

→ The sine of the angle b/w the element and the line joining point P to the element.

→ And inversely proportional to the square of the distance R b/w point P and the element.

Mathematically,

$$dH \propto \frac{I dL \sin \theta}{R^2}$$

$$dH = \frac{k I dL \sin \theta}{R^2}$$

where $k \rightarrow$ constant of proportionality

$$k = \frac{1}{4\pi}$$

$$dH = \frac{I dL \sin \theta}{4\pi R^2}$$

Vector form :

dL = magnitude of vector length dL

$a_R \rightarrow$ Unit vector in the direction from differential current element to point P.

Then from rule of cross product,

$$dL \times a_R = dL a_R \sin \theta = dL \sin \theta$$

$$\therefore dH = \frac{I dL \times a_R}{4\pi R^2}$$

$$a_R = \frac{R}{|R|}$$

$$\text{Hence } dH = \frac{I dL \times R}{4\pi R^3}$$

The entire conductor is made of elements. Hence to obtain total magnetic field intensity H .

$$\therefore H = \oint \frac{I dL \times \vec{a}_{R12}}{4\pi R_{12}^2}$$

The closed line integral is required to ensure that all the current elements are considered. This is because current can flow only in the closed path, provided by the closed circuit. If the current element is considered at point 1 and point P at point 2.

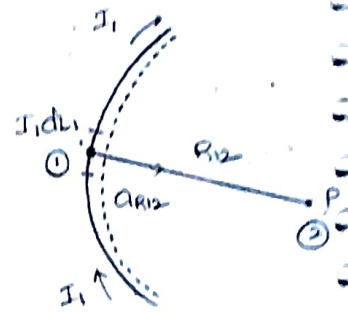


Fig. 2.3

$$dH_2 = \frac{I_1 dL_1 \times \vec{a}_{R12}}{4\pi R_{12}^2} \quad A/m$$

Where $I_1 \rightarrow$ current flowing through dL_1 at point 1

$dL_1 \rightarrow$ Differential vector length at point 1

$\vec{a}_{R12} \rightarrow$ Unit vector in the direction from element at point 1 to the point P at point 2

$$\vec{a}_{R12} = \frac{\vec{R}_{12}}{|\vec{R}_{12}|}$$

$$H_2 = \oint \frac{I_1 dL_1 \times \vec{a}_{R12}}{4\pi R_{12}^2} \quad A/m$$

This is called integral form of Biot-Savart Law.

Application of Biot-Savart Law

H due to Infinitely Long straight conductor

Consider an infinitely long straight conductor,

along z -axis. The current passing through

the conductor is a direct current of I amp.

The field intensity H at a point P is to be

calculated which is at a distance r from

the z -axis.

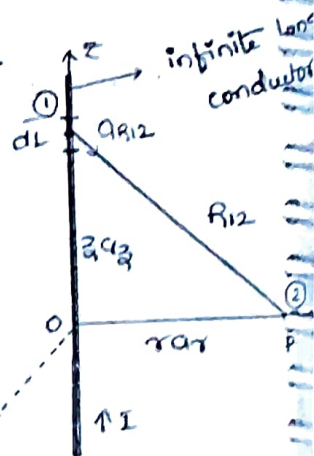


Fig. 2.4

consider small differential element at point 1, along the z -axis, at a distance z from origin. Let $r = r$

$$IdL = Idz a_z$$

$$R_{12} = -za_z + ra_r$$

$$a_{R12} = \frac{R_{12}}{|R_{12}|} = \frac{r a_r - z a_z}{\sqrt{r^2 + z^2}}$$

$$\therefore dL \times a_{R12} = \begin{vmatrix} a_r & a_\phi & a_z \\ 0 & 0 & dz \\ r & 0 & -z \end{vmatrix} = r dz a_\phi$$

$$\therefore IdL \times a_{R12} = \frac{I r dz a_\phi}{\sqrt{r^2 + z^2}}$$

According to Bio-t-Savart Law, dH at point 2 is

$$dH = \frac{IdL \times a_{R12}}{4\pi R_{12}^2} = \frac{I r dz a_\phi}{4\pi \sqrt{r^2 + z^2} [\sqrt{r^2 + z^2}]^2}$$

$$dH = \frac{I r dz a_\phi}{4\pi (r^2 + z^2)^{3/2}}$$

Thus total field intensity, H can be obtained by integrating dH over the entire length of the conductor.

$$H = \int_{z=-\infty}^{\infty} dH = \int_{-\infty}^{\infty} \frac{I r dz a_\phi}{4\pi (r^2 + z^2)^{3/2}}$$

$$\text{put } z = r \tan \theta, \quad dz = r \sec^2 \theta d\theta$$

$$z = -\infty, \theta = -\frac{\pi}{2} \text{ and } z = \infty, \theta = +\frac{\pi}{2}$$

$$H = \int_{-\pi/2}^{\pi/2} \frac{I r r \sec^2 \theta d\theta a_\phi}{4\pi (r^2 + r^2 \tan^2 \theta)^{3/2}}$$

$$= \int_{-\pi/2}^{\pi/2} \frac{I r^2 \sec^2 \theta d\theta a_\phi}{4\pi r^3 \sec^3 \theta} \quad \because 1 + \tan^2 \theta = \sec^2 \theta$$

$$= \int_{-\pi/2}^{\pi/2} \frac{I}{4\pi r} \cdot \frac{1}{\sec \theta} d\theta a_\phi = \frac{1}{4\pi r} \int_{-\pi/2}^{\pi/2} \cos \theta d\theta a_\phi$$

$$H = \frac{I}{4\pi r} \left[\sin\theta \right]_{-\pi/2}^{\pi/2} a\phi = \frac{I}{4\pi r} \left[\sin\frac{\pi}{2} - \sin\left(-\frac{\pi}{2}\right) \right] a\phi$$

$$H = \frac{I}{4\pi r} [1 - (-1)] a\phi = \frac{2I}{4\pi r} a\phi$$

$$H = \frac{I}{2\pi r} a\phi \text{ A/m}$$

Magnetic Flux Density :

The total magnetic lines of force i.e magnetic flux crossing a unit area in a plane at right angles to the direction of flux is called magnetic flux density. It is denoted as $\vec{B}$ and is a vector quantity. It is measured in weber per square metre (Wb/m^2), which is also called Tesla (T).

Relation between $\vec{B}$ and $\vec{H}$:

The B and H are related to each other through the property of the region in which current carrying conductor is placed. It is called permeability denoted as μ . For a free space, the permeability is denoted as μ_0 its value is $4\pi \times 10^{-7}$, the permeability μ is measured in (T/m), and $\mu = \mu_0 \mu_r$

$$B = \mu H$$

$$B = \mu_0 \mu_r H$$

For free space $B = \mu_0 H$

$$\therefore \mu_r = 1.$$

Ampere's Circuital Law ÷

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In the magnetostatics, the complex problems can be solved using a law called Ampere's Circuital Law or Ampere's work law.

The Ampere's circuital law states that, the line integral of magnetic field intensity H around a closed path is exactly equal to the direct current enclosed by that path.

Mathematically,

$$\oint H \cdot dL = I$$

Proof of Ampere's Circuital Law ÷

consider a long straight conductor carrying direct current I placed along z -axis.

consider a closed circular path of radius r which encloses the straight conductor -

carrying direct current I . the point P is

at a perpendicular distance r from the conductor. consider dL at point P which is in a_ϕ direction tangential to circular path at point P .

$$\therefore dL = r d\phi a_\phi$$

while H obtained at point P , from Biot - Savart Law due to infinitely long conductor is

$$H = \frac{I}{2\pi r} a_\phi$$

$$H \cdot dL = \frac{I}{2\pi r} a_\phi \cdot r d\phi a_\phi$$

$$= \frac{I}{2\pi r} r d\phi = \frac{I}{2\pi} d\phi$$

$$\therefore a_\phi \cdot a_\phi = 1$$

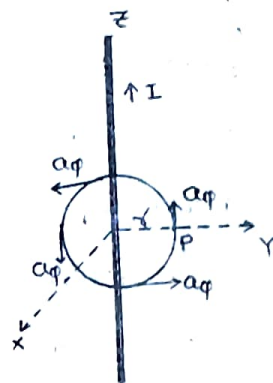


Fig. 2.5

Integrating $\vec{H} \cdot d\vec{L}$ Over the entire closed path,

$$\oint \vec{H} \cdot d\vec{L} = \int_0^{2\pi} \frac{I}{2\pi} d\phi = \frac{I}{2\pi} [\phi]_0^{2\pi} = I \frac{2\pi}{2\pi}$$

$$\boxed{\oint \vec{H} \cdot d\vec{L} = I} \rightarrow \text{current carried by conductor.}$$

Applications of Ampere's Circuital Law:

1) Due to infinitely long straight conductor:

consider an infinitely long straight conductor placed along z-axis, carrying a direct-current I . consider the amperian closed path, enclosing the conductor. consider

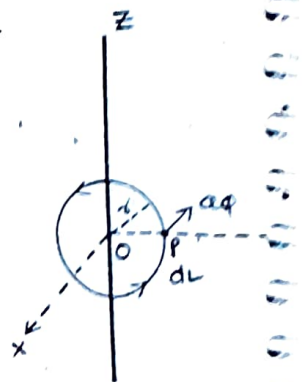


Fig. 2.6

point P on the closed path at which $\vec{H}$ is to be obtained. the radius of the path is r and hence P is at a perpendicular distance r from the conductor.

The magnitude of $\vec{H}$ depends on r and the direction is always tangential to the closed path. i.e. a_ϕ . so $\vec{H}$ has only component in a_ϕ direction say H_ϕ . consider elementary length dL at point P and in cylindrical co-ordinates it is $r d\phi$ in a_ϕ direction.

$$\vec{H} = H_\phi a_\phi \quad \text{and} \quad d\vec{L} = r d\phi a_\phi$$

$$\vec{H} \cdot d\vec{L} = H_\phi a_\phi \cdot r d\phi a_\phi = H_\phi r d\phi$$

$$\because a_\phi \cdot a_\phi = 1$$

According to Ampere's Circuital Law

$$\oint \vec{H} \cdot d\vec{L} = I$$

$$\int_{\phi=0}^{2\pi} H_\phi r d\phi = I$$

$$H_\phi r \int_0^{2\pi} d\phi = I$$

$$H_{\phi} r(2\pi) = I$$

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$$H_{\phi} = \frac{I}{2\pi r}$$

Hence $\vec{H}$ at point P is given by

$$\vec{H} = H_{\phi} a_{\phi} = \frac{I}{2\pi r} a_{\phi}$$

$\vec{H}$ Due to a co-axial cable :-

consider a co-axial cable. Its inner conductor is with radius a , carrying direct current I .

The outer conductor is in the form of - concentric cylinder whose inner radius is b and outer radius is c . This cable is placed

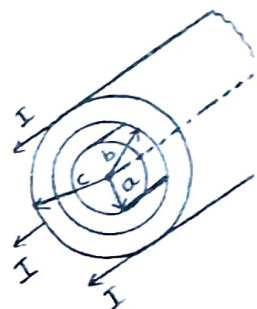


Fig. 2.7.1

along z -axis. The current I is uniformly distributed in the inner conductor. While $-I$ is uniformly distributed in the outer conductor.

Region 1: Within the inner conductor $r < a$. Consider a closed path having radius $r < a$. Hence it encloses only part of the conductor.

The area of cross-section enclosed is πr^2

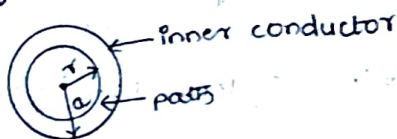


Fig. 2.8.2

The total current flowing is I through the area πa^2 . Hence the current enclosed by the closed path is

$$I' = \frac{\pi r^2}{\pi a^2} I = \frac{r^2}{a^2} I$$

The $\vec{H}$ is again only in a_{ϕ} direction and depends only on r .

$$H = H_\phi a_\phi$$

Consider dL in the a_ϕ direction which is $r d\phi$

$$dL = r d\phi a_\phi$$

$$H \cdot dL = H_\phi a_\phi \cdot r d\phi a_\phi = H_\phi r d\phi$$

According to Ampere's circuital Law

$$\oint H \cdot dL = I'$$

$$\oint H_\phi r d\phi = \frac{r^2}{a^2} I$$

$$\int_0^{2\pi} H_\phi r d\phi = \frac{r^2}{a^2} I$$

$$H_\phi r [2\pi] = \frac{r^2}{a^2} I$$

$$H_\phi = \frac{r^2}{2\pi r a^2} I = \frac{r}{2\pi a^2} I$$

$$\boxed{H = \frac{I r}{2\pi a^2} a_\phi} \quad \text{A/m}$$

$r < a$ within core

Region 2: Within $a < r < b$ consider a circular path which encloses the inner conductor carrying direct current. This is the case of infinitely long conductor along z . Hence H in this region is

$$\boxed{H = \frac{I}{2\pi r} a_\phi} \quad \text{A/m}$$

Region 3: Within outer conductor $b < r < c$

consider the closed path. The current enclosed by the closed path is only the part of the current $-I$, in the outer conductor. The total current $-I$ is flowing through the cross section $\pi(c^2 - b^2)$ while the closed path encloses the cross section $\pi(r^2 - b^2)$.

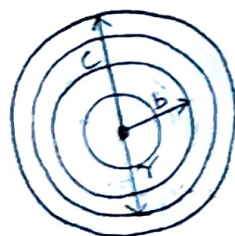


Fig. 2.7.3

the closed path encloses

Hence the current enclosed by the closed path of outer conductor is

$$I' = \frac{\pi (r^2 - b^2)}{\pi (c^2 - b^2)} (-I) = - \frac{(r^2 - b^2)}{c^2 - b^2} I.$$

$\therefore I'' = I =$ current in inner conductor enclosed.

Total current enclosed by the closed path is

$$\begin{aligned} I_{enc} &= I' + I'' = - \frac{(r^2 - b^2)}{c^2 - b^2} (I + I) \\ &= I \left[1 - \frac{r^2 - b^2}{c^2 - b^2} \right] = I \left[\frac{c^2 - r^2}{c^2 - b^2} \right] \end{aligned}$$

According to ampere's circuital law

$$\oint \vec{H} \cdot d\vec{L} = I_{enc}$$

Now $\vec{H}$ is again in $\hat{a}_\phi$ direction only and is a f.f. of r .

$$H = H_\phi \hat{a}_\phi \quad \text{and} \quad dL = r d\phi \hat{a}_\phi$$

$$\vec{H} \cdot d\vec{L} = H_\phi \hat{a}_\phi \cdot r d\phi \hat{a}_\phi = H_\phi r d\phi$$

$$\int_{\phi=0}^{2\pi} H_\phi r d\phi = I_{enc}$$

$$H_\phi r [2\pi] = I \left[\frac{c^2 - r^2}{c^2 - b^2} \right]$$

$$H_\phi = \frac{I}{2\pi r} \left[\frac{c^2 - b^2}{c^2 - r^2} \right]$$

$$\boxed{H = H_\phi \hat{a}_\phi = \frac{I}{2\pi r} \left[\frac{c^2 - b^2}{c^2 - r^2} \right] \hat{a}_\phi \quad \text{A/m} \quad \dots b < r < c}$$

Region-4 Outside the cable, $r > c$

consider the closed path with $r > c$ such that it encloses both the conductors i.e. both currents $+I$ and $-I$.

Thus the total current enclosed is

$$I_{enc} = +I - I = 0 \text{ A}$$

$$\oint \vec{H} \cdot d\vec{L} = 0$$

$$H = 0$$

From Amperes circuital Law

$$\oint \mathbf{H} \cdot d\mathbf{L} = I$$

$$\text{But } I = \int \mathbf{J} \cdot d\mathbf{s}$$

$$\oint \mathbf{H} \cdot d\mathbf{L} = \int \mathbf{J} \cdot d\mathbf{s}$$

↓

Apply Stokes theorem

$$\int \nabla \times \mathbf{H} \cdot d\mathbf{s} = \int \mathbf{J} \cdot d\mathbf{s}$$

$$\therefore \boxed{\nabla \times \mathbf{H} = \mathbf{J}} \rightarrow \text{This is called Maxwell's eq.}^{\text{II}}$$

If the flux passing through the unit area is not exactly at right angles to the plane consisting the area but making some angle with the plane then the flux ϕ crossing the area is given by

$$\phi = \int_S \mathbf{B} \cdot d\mathbf{s} \quad \text{webers}$$

$\phi \rightarrow$ Magnetic flux in webers

$\mathbf{B} \rightarrow$ Magnetic flux density

$d\mathbf{s} \rightarrow$ open surface through which flux is passing

Now consider a closed surface which is defining a certain volume. The magnetic flux lines are always exist in the form of closed loop. The single magnetic pole cannot exist like a single isolated electric charge. No magnetic flux can reside in a closed surface. Hence the integral $\mathbf{B} \cdot d\mathbf{s}$ evaluated over a closed surface is zero.

$$\oint_S \mathbf{B} \cdot d\mathbf{s} = 0 \rightarrow \text{Gauss's Law in integral form}$$

Applying divergence theorem

$$\oint_S \mathbf{B} \cdot d\mathbf{s} = \int_{V_1} \nabla \cdot \mathbf{B} \, dv = 0 \rightarrow \boxed{\nabla \cdot \mathbf{B} = 0}$$

The divergence of magnetic flux density is zero. This is another Maxwell's eq. ^{III}

Magnetic Scalar and Vector Potentials :-

In electrostatics, it is seen that there exists a scalar electric potential V which is related to the electric field intensity E as $E = -\nabla V$.

In case of magnetic fields there are two types of potential which can be defined.

→ The scalar magnetic potential denoted as V_m .

→ The vector magnetic potential denoted as $\vec{A}$.

To define scalar and vector magnetic potentials, let us use two vector identities which are listed as the properties of curl.

$$\nabla \times \nabla V_m = 0, \quad V_m \rightarrow \text{Scalar} \quad \text{--- (1)}$$

$$\nabla \cdot (\nabla \times \vec{A}) = 0, \quad \vec{A} \rightarrow \text{Vector} \quad \text{--- (2)}$$

Every scalar V and vector $\vec{A}$ must satisfy these identities.

Scalar Magnetic Potential :-

If V_m is the scalar magnetic potential, then it must satisfy the eqⁿ (1)

$$\nabla \times \nabla V_m = 0 \quad \text{--- (3)}$$

But the scalar magnetic potential is related to the magnetic field intensity $\vec{H}$ as

$$\vec{H} = -\nabla V_m \quad \text{--- (4)}$$

using in eqⁿ (3)

$$\nabla \times (-\vec{H}) = 0 \quad \text{i.e. } \nabla \times \vec{H} = 0$$

$$\text{i.e. } \vec{J} = 0.$$

$$\text{But } \nabla \times \vec{H} = \vec{J}$$

Thus scalar magnetic potential V_m can be defined for source free region where $\vec{J}$ i.e. current density is zero.

$$\vec{H} = -\nabla V_m \quad \text{only for } \vec{J} = 0$$

$\therefore$ Magnetic scalar potential can be expressed in terms

$$V_{m(a,b)} = -\int_a^b \vec{H} \cdot d\vec{L}$$

Laplace's Equation for scalar

$$\oint B \cdot ds = 0$$

Using Divergence theorem

$$\oint B \cdot ds = \int_{Vol} (\nabla \cdot B) dv = 0$$

$$\nabla \cdot B = 0$$

$$\nabla \cdot (\mu_0 H) = 0$$

but $\mu_0 \neq 0$

$$\nabla \cdot H = 0$$

$$\nabla \cdot (-\nabla V_m) = 0$$

using $H = -\nabla V_m$

$$\boxed{\nabla^2 V_m = 0}$$

this is Laplace's eqⁿ for scalar magnetic potential.
this is similar to the Laplace's eqⁿ for scalar electric potential $\nabla^2 V = 0$.

Vector Magnetic Potential

The vector magnetic potential is denoted as $\vec{A}$ and measured in Wb/m. It has to satisfy eqⁿ (2) that divergence of a curl of a vector is always zero.

$$\nabla \cdot (\nabla \times \vec{A}) = 0$$

$\vec{A} \rightarrow$ Vector Magnetic Potential

$$\text{But } \nabla \cdot B = 0$$

compare above eqⁿs

$$B = \nabla \times \vec{A}$$

Thus curl of vector magnetic potential is the flux density.

$$\nabla \times H = J$$

$$\nabla \times \frac{B}{\mu_0} = J$$

$$\because B = \mu_0 H$$

$$\nabla \times B = \mu_0 J$$

$$\because B = \nabla \times \vec{A}$$

$$\nabla \times \nabla \times \vec{A} = \mu_0 J$$

Using vector identity to express left hand side

$$\therefore \nabla (\nabla \cdot \vec{A}) - \nabla^2 \vec{A} = \mu_0 J$$

$$J = \frac{1}{\mu_0} [\nabla \times \nabla \times \vec{A}] = \frac{1}{\mu_0} [\nabla (\nabla \cdot \vec{A}) - \nabla^2 \vec{A}]$$

Poisson's Equation for Magnetic Field :-

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For a vector magnetic potential $\vec{A}$, its curl is defined as $\nabla \times \vec{A} = \vec{B}$ which is known. But to complete define $\vec{A}$ its divergence must be known. Assume that $\nabla \cdot \vec{A}$, the divergence of $\vec{A}$ is zero.

$$\therefore \vec{J} = \frac{1}{\mu_0} [-\nabla^2 \vec{A}]$$

$$\boxed{\nabla^2 \vec{A} = -\mu_0 \vec{J}}$$

This is the Poisson's eqn. for magnetostatic fields.

$\vec{A}$ due to Differential current element :-

consider the differential element $d\vec{l}$ carrying current I . Then according to Biot-Savart law the vector magnetic potential $\vec{A}$ at a distance R from the differential current element is given by

$$\vec{A} = \oint \frac{\mu_0 I d\vec{l}}{4\pi R} \text{ Wb/m}$$

For the distributed current source, $I d\vec{l}$ can be replaced by $\vec{K} ds$ where $\vec{K}$ is surface current density.

$$\vec{A} = \oint_s \frac{\mu_0 \vec{K} ds}{4\pi R} \text{ Wb/m}$$

The line integral becomes a surface integral. If the volume current density $\vec{J}$ is given in A/m^2 then $I d\vec{l}$ can be replaced by $\vec{J} dv$ where dv is differential volume element.

$$\vec{A} = \oint_{\text{Vol}} \frac{\mu_0 \vec{J} dv}{4\pi R} \text{ Wb/m}$$

Inductance :-

When a coil with N turns, carrying current I , the flux is produced by it. This flux links with each turn of the coil. Thus total flux linkage of the coil having N turns $N\Phi$.

The flux linked with the coil is proportional to the current I flowing through it. the ratio of total flux linkage to the current producing that flux is called inductance denoted by L . It is measured in henry (H)

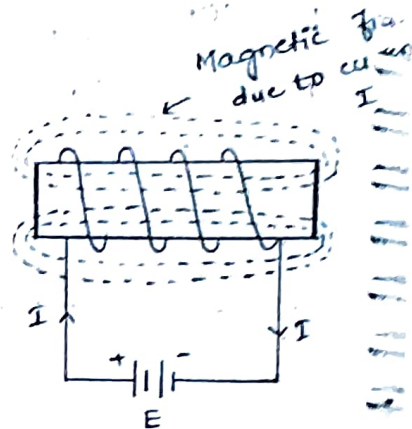


Fig. 2.8

$$L = \frac{N\Phi}{I}$$

In general, inductance is also referred as self inductance as the flux produced by the current flowing through the coil links with the coil itself.

Mutual Inductance:

consider two magnetic circuits with two different coils carrying two different currents.

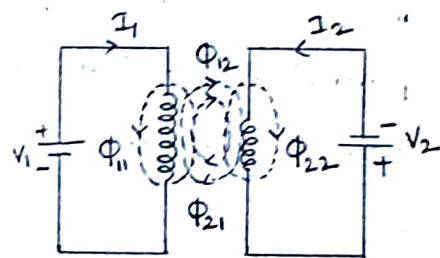


Fig. 2.9

Let coil 1 of N_1 turns with inductance L_1 carries current I_1 . The magnetic flux produced by current I_1 flowing through coil 1 is represented by Φ_{11} .

Let coil 2 of N_2 turns with inductance L_2 carries current I_2 . then the magnetic flux produced by current I_2 flowing through coil 2 is represented by Φ_{22} .

The flux produced in coil 1 also links with coil 2 produces another flux represented by Φ_{12} . Similarly the flux produced in coil 2 links with coil 1 producing flux - represented by Φ_{21} . The total flux linkage of the second coil due to the flux produced by current I_1 in first coil is $N_2\Phi_{12}$. Similarly the total flux linkage of the first coil due to the flux produced by current I_2 in the second coil, is $N_1\Phi_{21}$.

With respect to the flux linkage, the mutual inductance b/w the two coils is defined as the ratio of flux linkage of one coil to the current in other coil. Thus the mutual inductance b/w circuit 1 and circuit 2 is given by

$$M_{12} = \frac{N_2 \Phi_{12}}{I_1}$$

Note that Φ_{12} is the flux produced by I_1 which links the second coil carrying current I_2 .

Similarly the mutual inductance b/w circuit 1 and 2 can be represented as

$$M_{21} = \frac{N_1 \Phi_{21}}{I_2}$$

Obviously, here Φ_{21} is the flux produced by I_2 which links the first coil carrying current I_1 .

If the medium surrounding the magnetic circuits is linear

$$M_{12} = M_{21}$$

Applications of Inductance :-

Inductance of a solenoid :-

Consider a solenoid of N turns. Let the current flowing through the solenoid be I ampere. Let the length of the solenoid be L and the cross-sectional area be A .

From the results obtained, the field intensity inside the solenoid is given by

$$H = \frac{NI}{L}$$

The total flux linkage is given by,

$$\begin{aligned} \text{Total flux linkage} &= N\Phi = NB A \\ &= N(\mu H) A \end{aligned}$$

$$\begin{aligned} \text{Total flux linkage} &= \mu N H A \\ &= \mu N \left[\frac{NI}{L} \right] A \\ &= \frac{\mu N^2 I A}{L} \end{aligned}$$

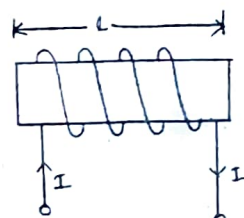


Fig. 2.9.1

Thus the inductance of a solenoid is given by

$$L = \frac{\text{Total Flux Linkage}}{\text{Total current}}$$

$$L = \frac{\mu N^2 I A}{L(I)}$$

$$L = \frac{\mu N^2 A}{L}$$

Inductance of a Toroid :-

consider a toroidal ring with N turns and carrying current I . Let the radius of the toroid be R .

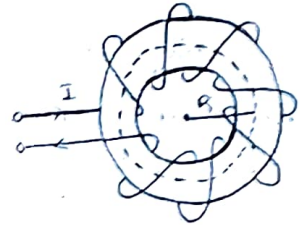


Fig. 2.9.2

The magnetic flux density inside a toroidal ring is given by

$$B = \frac{\mu N I}{2\pi R}$$

The total flux linkage of a toroidal ring having N turns is given by

$$\text{Total Flux linkage} = N\Phi$$

$$\text{But } \Phi = B \cdot A$$

where $A \rightarrow$ Area of cross-section of a ring

$$\therefore \text{Total Flux linkage} = N B A$$

$$= N \cdot \frac{\mu N I}{2\pi R} \cdot A$$

$$= \frac{\mu N^2 I A}{2\pi R}$$

The inductance of a toroid is given by

$$L = \frac{\text{Total Flux Linkage}}{\text{Total current}}$$

$$L = \frac{\mu N^2 I A}{(2\pi R) I}$$

$$L = \frac{\mu N^2 A}{2\pi R}$$

$$A = \pi r^2$$

Biot-Savart Law :

$$dH = \frac{I dL \times q_B}{4\pi R^3} \text{ A/m}$$

Biot-Savart Law in terms of different charge distributions :

$$\text{Surface current } H = \int_S \frac{K \times q_B}{4\pi R^2} ds \text{ A/m}$$

$K \rightarrow$ Surface current density in A/m

$$\text{Volume current } H = \int_V \frac{J \times q_B}{4\pi R^2} dv \text{ A/m}$$

$J \rightarrow$ Volume current density in A/m²

Application of Biot-Savart Law :

Infinitely long straight conductor along z-axis

$$H = \frac{I}{2\pi r} a_\phi \text{ A/m}$$

Ampere's Circuital Law and Applications :

$$\oint H \cdot dL = I \text{ (Current enclosed)}$$

Infinite long straight conductor carrying current I,

$$H = \frac{I}{2\pi r} a_\phi \text{ A/m}$$

co-axial cable

inner conductor radius = a

Inner radius of outer = b

outer radius of outer = c

Radial distance from centre = r

$$H = \begin{cases} \frac{I r}{2\pi a^2} a_\phi & r < a \\ \frac{I}{2\pi r} a_\phi & a < r < b \\ \frac{I}{2\pi r} \left[\frac{c^2 - r^2}{c^2 - b^2} \right] a_\phi & b < r < c \\ 0 & r > c \end{cases}$$

Point Form of Ampere's Circuital Law :

$$\nabla \times H = J$$

Inductance of a co-axial cable

consider a co-axial cable with inner conductor radius a and outer conductor radius b . Let the current through the co-axial cable be I .

For the co-axial cable the field intensity at any point b/w inner and outer conductor is given by

$$H = \frac{I}{2\pi r} \quad \text{where } a \leq r \leq b$$

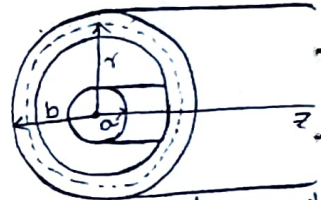


Fig. 2.9.3

Now assume that the axis of the cable is along z -axis. the magnetic flux density will be in radial plane extending from $r=a$ to $r=b$ and $z=0$ to $z=d$.

Now assume that the axis of the cable is along z -axis

$$B = \frac{\mu I}{2\pi r} a_\phi$$

The total magnetic flux is given by

$$\Phi = \int_S B \cdot ds$$

$$\text{let } r = r$$

$$ds = dr dz a_\phi$$

$$\Phi = \int_{z=0}^d \int_{r=a}^b \frac{\mu I}{2\pi r} a_\phi \cdot dr dz a_\phi$$

$$\Phi = \frac{\mu I}{2\pi} [z]_0^d [\ln r]_a^b$$

$$\Phi = \frac{\mu I d}{2\pi} \ln\left(\frac{b}{a}\right)$$

The inductance of a co-axial cable is given by

$$L = \frac{\text{Total flux linkage}}{\text{Total current}}$$

$$L = \frac{\frac{\mu I d}{2\pi} \ln\left(\frac{b}{a}\right)}{I}$$

$$L = \frac{\mu d}{2\pi} \ln\left(\frac{b}{a}\right)$$

Magnetic Flux and Flux density:

$$\Phi = \int_S \mathbf{B} \cdot d\mathbf{s} \quad \text{Wb}$$

$\oint_S \mathbf{B} \cdot d\mathbf{s} = 0 \rightarrow$ Over the closed surface Gauss's Law in integral form

$\nabla \cdot \mathbf{B} = 0 \rightarrow$ Gauss's Law in point form.

Maxwell's Equation:

$\nabla \times \mathbf{H} = \mathbf{J} \rightarrow$ Ampere's circuital Law

$\nabla \cdot \mathbf{B} = 0 \rightarrow$ from Gauss's Law.

Magnetic Scalar and Vector Potentials:

$V_m \rightarrow$ scalar magnetic potential

$$\nabla \times \nabla V_m = 0$$

$$\mathbf{H} = -\nabla V_m$$

$$V_m = - \int_b^a \mathbf{H} \cdot d\mathbf{L} \rightarrow \text{specified path}$$

$$\nabla^2 V_m = 0$$

$\mathbf{A} \rightarrow$ Vector magnetic Potential

$$\nabla \cdot (\nabla \times \mathbf{A}) = 0$$

$$\mathbf{B} = \nabla \times \mathbf{A}$$

$$\nabla \times \nabla \times \mathbf{A} = \mu_0 \mathbf{J}$$

$$\mathbf{J} = \frac{1}{\mu_0} [\nabla(\nabla \cdot \mathbf{A}) - \nabla^2 \mathbf{A}]$$

$\nabla^2 \mathbf{A} = -\mu_0 \mathbf{J} \rightarrow$ is Poisson's equation

$\bar{A}$ Due to differential current element

$$\bar{A} = \oint_L \frac{\mu_0 I d\mathbf{L}}{4\pi R} \quad \text{Wb/m}$$

$$\mathbf{A} = \int_S \frac{\mu_0 \mathbf{K} d\mathbf{s}}{4\pi R} \quad \text{Wb/m} \rightarrow \text{surface current}$$

$$\mathbf{A} = \int_{vol} \frac{\mu_0 \mathbf{J} d\mathbf{v}}{4\pi R} \quad \text{Wb/m} \rightarrow \text{volume current}$$

Prob: Find the magnetic field intensity at the origin due to a current element $5\pi a_3 \mu A \cdot m$ at the point $(2, -5, 0)$ in free space.

Solu: Given $I dL = 5\pi a_3 \mu A \cdot m$, point $P = (2, -5, 0)$

Distance of P from the origin, $R = \sqrt{2^2 + 5^2} = \sqrt{29}$

Directed line segment from P to O , $R = -2a_x + 5a_y$

Let $a_R \rightarrow$ unit vector along PO

$$a_R = \frac{-2}{\sqrt{29}} a_x + \frac{5}{\sqrt{29}} a_y$$

$$a_R = 0.371 a_x + 0.92 a_y$$

Let us next obtain the cross product $(I dL \times a_R)$

Given $I dL = 5\pi a_3$

$$I dL \times a_R = \begin{vmatrix} a_x & a_y & a_z \\ 0 & 0 & 5\pi \\ 0.37 & 0.92 & 0 \end{vmatrix}$$

$$I dL \times a_R = -14.58 a_x + 5.83 a_y + 0 a_z$$

$$dH = \frac{I dL \times a_R}{4\pi R^2}$$

$$dH = \frac{(-14.58 a_x + 5.83 a_y) \times 10^{-6}}{4\pi \times 29} \text{ A/m}$$

$$dH = (-40 a_x + 16 a_y) \text{ nA/m}$$

Prob: In the region $0 < r < 0.5 \text{ m}$, in cylindrical co-ordinates the current density is $J = 4.5 e^{-2r} a_3 \text{ A/m}^2$. And $J = 0$ elsewhere use Amperes circuital law to find H .

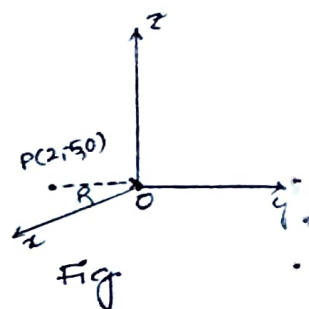
Solu: The current from current density is given by

$$I = \oint J \cdot ds$$

$ds = r dr d\phi a_3$ normal to a_3 as J is in a_3

$$I = \int_{\phi=0}^{2\pi} \int_{r=0}^r 4.5 e^{-2r} a_3 \cdot r dr d\phi a_3$$

$$I = 4.5 \int_{\phi=0}^{2\pi} \int_{r=0}^r r e^{-2r} dr d\phi$$



Using integration by parts

$$I = 4.5 \int_{\phi=0}^{2\pi} d\phi \left\{ r \int_0^r e^{-2r} dr - \int_1^r e^{-2r} dr \right\}$$

$$I = 4.5 (2\pi) \left\{ \frac{r e^{-2r}}{-2} - \int \frac{e^{-2r}}{-2} dr \right\}_0^r$$

$$I = 9\pi \left\{ \frac{r e^{-2r}}{-2} + \frac{1}{2} \frac{e^{-2r}}{-2} \right\}_0^r$$

$$I = 9\pi \left\{ -\frac{r e^{-2r}}{2} - \frac{1}{4} e^{-2r} + \frac{1}{4} \right\}$$

for $r = 0.5$

$$I = 1.8676 \text{ A}$$

Consider a closed path with $r \geq 0.5$ such that the enclosed current I is 1.867 A .

According to Ampere's Circuital Law

$$\oint H \cdot dL = I$$

$$\int_{\phi=0}^{2\pi} H_{\phi} r d\phi = I$$

$$\therefore H = H_{\phi} a_{\phi}, dL = r d\phi a_{\phi}$$

$$a_{\phi} \cdot a_{\phi} = 1$$

$$2\pi r H_{\phi} = 1.8676$$

$$H_{\phi} = \frac{1.8676}{2\pi r}$$

$$H_{\phi} = \frac{0.2972}{r} a_{\phi} \text{ A/m for } r \geq 0.5 \text{ m}$$

Prob: A radial field $H = \frac{2.39 \times 10^6}{r} \cos \phi$ A/m exists in ϕ space. Find the magnetic flux crossing the surface defined by $0 \leq \phi \leq \frac{\pi}{4}$ and $0 \leq z \leq 1$.

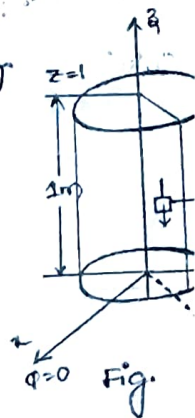
Solu: The portion of the cylinder is shown in fig the flux crossing the given surface is given by

$$\phi = \int_S B \cdot ds$$

ds normal to $\bar{a}_r$ direction is $ds = r d\phi dz \bar{a}_r$

$$\phi = \int_S \mu_0 \bar{H} \cdot ds$$

$$B = \mu_0 H$$



$$\Phi = \mu_0 \int_0^1 \int_0^{\pi/4} \frac{2.39 \times 10^6}{r} \cos \phi \, a_r \cdot r \, d\phi \, dz \, a_z$$

$$\Phi = \mu_0 \int_{z=0}^1 \int_{\phi=0}^{\pi/4} 2.39 \times 10^6 \cos \phi \, d\phi \, dz$$

$$\Phi = 2.39 \times 10^6 \mu_0 \left[\sin \phi \right]_0^{\pi/4} \left[z \right]_0^1$$

$$\Phi = 2.1236 \text{ Wb}$$

Prob: Find the flux passing the portion of the plane $\phi = \frac{\pi}{4}$ defined by $0.01 \leq r \leq 0.05 \text{ m}$ and $0 \leq z \leq 2 \text{ m}$. A current filament of 2.5 A is along the z -axis in the a_z direction in free space.

Solu: Due to current carrying conductor in free space along z axis, $\vec{H}$ is given by

$$\vec{H} = \frac{I}{2\pi r} a_\phi$$

$$\vec{H} = \frac{2.5}{2\pi r} a_\phi$$

$$\vec{H} = \frac{0.3978}{r} a_\phi \text{ A/m}$$

$$\vec{B} = \mu_0 \vec{H} = \frac{4\pi \times 10^{-7} \times 0.397}{r} a_\phi$$

$$\vec{B} = \frac{5 \times 10^{-7}}{r} a_\phi \text{ Wb/m}^2$$

The flux crossing the surface is given by

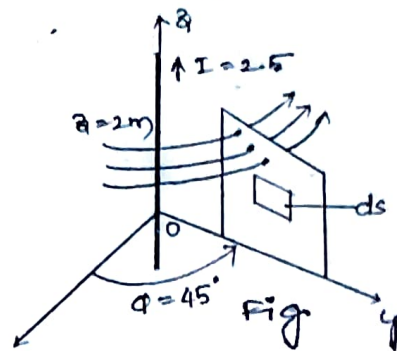
$$\Phi = \int_S \vec{B} \cdot d\vec{s}$$

Now $d\vec{s} = dr \, dz \, a_\phi$ normal to a_ϕ direction

$$\Phi = \int_{z=0}^2 \int_{r=0.01}^{0.05} \frac{5 \times 10^{-7}}{r} a_\phi \cdot dr \, dz \, a_\phi$$

$$\Phi = \int_{z=0}^2 \int_{r=0.01}^{0.05} \frac{5 \times 10^{-7}}{r} dr \, dz = 5 \times 10^{-7} \left[\ln r \right]_{0.01}^{0.05} \left[z \right]_0^2$$

$$\Phi = 1.609 \mu\text{Wb}$$



Faraday's Law :-

Faraday's Law states that the induced e.m.f is equal to the time rate of change of magnetic flux linking with the closed circuit.

Faraday's Law can be stated as,

$$e = -N \frac{d\phi}{dt} \text{ volts.}$$

where $N \rightarrow$ Number of turns in the circuit.

$e \rightarrow$ induced e.m.f.

Let us assume single turn circuit i.e. $N=1$, then Faraday's Law can be stated as

$$e = - \frac{d\phi}{dt} \text{ volts.}$$

The Minus sign indicates that the direction of the induced e.m.f is such that to produce a current which will produce a magnetic field which will oppose the original field.

Let us consider Faraday's Law. the induced e.m.f is a scalar quantity measured in volts. thus the induced e.m.f is given by,

$$e = \oint \vec{E} \cdot d\vec{L}$$

The Magnetic flux ϕ passing through a specified area is given by

$$\phi = \int_S \vec{B} \cdot d\vec{s}$$

$B \rightarrow$ Magnetic flux density.

From above equations

$$e = - \frac{d}{dt} \int_S \vec{B} \cdot d\vec{s}$$

$$e = \oint \vec{E} \cdot d\vec{L} = - \frac{d}{dt} \int_S \vec{B} \cdot d\vec{s}$$

the closed circuit in which e.m.f. is induced is stationary and the magnetic flux is sinusoidally varying with time. From the equation it is clear that the magnetic flux ϕ is the only quantity varying with time. Hence we can write

$$\oint \vec{E} \cdot d\vec{L} = - \int_s \frac{\partial \vec{B}}{\partial t} \cdot d\vec{s}$$

this is similar to transformer action and e.m.f. is called transformer e.m.f. Using Stokes's theorem, a line integral can be converted to the surface integral as

$$\oint_s (\nabla \times \vec{E}) \cdot d\vec{s} = - \int_s \frac{\partial \vec{B}}{\partial t} \cdot d\vec{s}$$

Assuming that both the surface integrals taken over identical surfaces.

$$(\nabla \times \vec{E}) \cdot d\vec{s} = - \frac{\partial \vec{B}}{\partial t} \cdot d\vec{s}$$

Hence finally

$$\boxed{\nabla \times \vec{E} = - \frac{\partial \vec{B}}{\partial t}}$$

Inconsistency of Ampere's circuital law (or) Modification of Ampere's Law :-

For a static electromagnetic fields, according to Ampere's circuital law,

$$\nabla \times \vec{H} = \vec{J} \quad \text{--- (a)}$$

Taking divergence on both sides

$$\nabla \cdot (\nabla \times \vec{H}) = \nabla \cdot \vec{J}$$

But according to vector identity, divergence of the curl of any vector field is zero. Hence

$$\nabla \cdot (\nabla \times \vec{H}) = \nabla \cdot \vec{J} = 0 \quad \text{--- (b)}$$

But the equation of continuity is given by

$$\nabla \cdot \mathbf{J} = -\frac{\partial \rho_v}{\partial t}$$

When $\frac{\partial \rho_v}{\partial t} = 0$, then only eqn (1) become true. so the eqns are not compatible for time varying fields. therefore eqn (1) needs to be modified so that it is valid for time varying fields.

Let $\nabla \times \mathbf{H} = \mathbf{J} + \mathbf{G}$ where $\mathbf{G}$ is an unknown quantity which requires to be evaluated.

Taking divergence on both sides

$$\nabla \cdot (\nabla \times \mathbf{H}) = 0 = \nabla \cdot (\mathbf{J} + \mathbf{G})$$

$$= \nabla \cdot \mathbf{J} + \nabla \cdot \mathbf{G}$$

$$\nabla \cdot \mathbf{G} = -\nabla \cdot \mathbf{J}$$

$$\text{But } \nabla \cdot \mathbf{J} = -\frac{\partial \rho_v}{\partial t} \text{ (continuity eqn)}$$

$$\nabla \cdot \mathbf{G} = \frac{\partial \rho_v}{\partial t}$$

We have $\nabla \cdot \mathbf{D} = \rho_v$, putting $\rho_v = \nabla \cdot \mathbf{D}$ we get

$$\nabla \cdot \mathbf{G} = \frac{\partial}{\partial t} (\nabla \cdot \mathbf{D}) = \nabla \cdot \frac{\partial \mathbf{D}}{\partial t}$$

Since $\nabla \cdot \mathbf{G} = \nabla \cdot \frac{\partial \mathbf{D}}{\partial t}$, it is obvious that $\mathbf{G} = \frac{\partial \mathbf{D}}{\partial t}$

$$\nabla \times \mathbf{H} = \mathbf{J} + \mathbf{G} = \mathbf{J} + \frac{\partial \mathbf{D}}{\partial t}$$

$$\boxed{\nabla \times \mathbf{H} = \mathbf{J} + \frac{\partial \mathbf{D}}{\partial t}}$$

The above expression $\mathbf{J} \rightarrow$ conduction current density.

$\frac{\partial \mathbf{D}}{\partial t} \rightarrow$ displacement current density.

Non-time varying Maxwell's equations in Point Form (or) Differential

Form :-

- $\nabla \cdot D = \rho_v$ derived from Gauss's law for electric field.
- $\nabla \times E = 0$ derived from Potential
- $\nabla \times H = J$ derived from Ampere's Law
- $\nabla \cdot B = 0$ derived from Gauss's Law for magnetic field.

Word statement :-

$$\nabla \cdot D = \rho_v \quad \&$$

Non-time varying Maxwell's eqns in integral form :-

- $\oint_S D \cdot ds = \int_{vol} \rho_v dv$
- $\oint E \cdot dL = 0$
- $\oint H \cdot dL = \int_S J \cdot ds$
- $\oint B \cdot ds = 0$

Word statement :-

$$\nabla \cdot D = \rho_v \quad (\text{or}) \quad \oint_S D \cdot ds = \int_{vol} \rho_v dv$$

The total electric flux passing through any closed surface is equal to the total charge in the volume enclosed by the closed surface.

$$\nabla \times E = 0 \quad (\text{or}) \quad \oint E \cdot dL = 0$$

The electromotive force (e.m.f) around a closed path is equal to zero

$$\oint H \cdot dL = \int_S J \cdot ds \quad (\text{or}) \quad \nabla \times H = J$$

The magnetomotive force (m.m.f) around a closed path is equal to the surface integral of conduction current density over the surface bounded by the closed path.

$$\nabla \cdot B = 0 \quad (\text{or}) \quad \oint B \cdot ds = 0$$

The net magnetic flux emerging from a closed surface is zero.

Time varying Maxwell's eqs in point form (or) differential form

- $\nabla \cdot D = \rho_v$ derived from Gauss's Law for electric field.
- $\nabla \times E = -\frac{\partial B}{\partial t}$ derived from Faraday's Law of electromagnetic induction.
- $\nabla \times H = J + \frac{\partial D}{\partial t}$ derived from Ampere's circuital law.
- $\nabla \cdot B = 0$ derived from Gauss's Law for Magnetic field.

Time varying Maxwell's eqs in Integral form :-

$$\oint_S D \cdot ds = \int_{\text{Vol}} \rho_v dv$$

$$\oint E \cdot dL = - \int_S \frac{\partial B}{\partial t} \cdot ds$$

$$\oint H \cdot dL = \int_S (J + \frac{\partial D}{\partial t}) \cdot ds$$

$$\oint B \cdot ds = 0$$

Word statement :-

$$\nabla \cdot D = \rho_v \quad (\text{or}) \quad \oint_S D \cdot ds = \int_{\text{Vol}} \rho_v dv$$

The total electric flux passing through any closed surface is equal to the total charge in the volume enclosed by the closed surface.

$$\nabla \times E = -\frac{\partial B}{\partial t} \quad (\text{or}) \quad \oint E \cdot dL = - \int_S \frac{\partial B}{\partial t} \cdot ds$$

The electromotive force (or e.m.f) around a closed path is equal to the negative of the surface integral of the time rate of change of the magnetic flux density over the surface bounded by the closed path.

$$\nabla \times H = J + \frac{\partial D}{\partial t} \quad (\text{or}) \quad \oint H \cdot dL = \int \left(J + \frac{\partial D}{\partial t} \right) \cdot dS$$

The magnetomotive force (or m.m.f) around a closed path is equal to the surface integral of conduction current density and displacement current density over the surface bounded by the closed path.

$$\nabla \cdot B = 0 \quad (\text{or}) \quad \oint B \cdot dS = 0$$

The net magnetic flux emerging from a closed surface is zero.

Maxwell's equations for free space :-

Let us consider free space as a medium in which fields are present. Free space is a non-conducting medium in which volume charge density ρ_r is zero and conductivity σ is also zero. The Maxwell's equations in the free space are

Point form :-

$$\nabla \cdot D = 0$$

$$\nabla \times E = - \frac{\partial B}{\partial t}$$

$$\nabla \times H = \frac{\partial D}{\partial t}$$

$$\nabla \cdot B = 0$$

Integral form :-

$$\oint_S D \cdot dS = 0$$

$$\oint E \cdot dL = - \int_S \frac{\partial B}{\partial t} \cdot dS$$

$$\oint H \cdot dL = \int_S \frac{\partial D}{\partial t} \cdot dS$$

$$\oint_S B \cdot dS = 0$$

Maxwell's Equations for good conductor :-

It is clear from the type of the medium, that the conductivity is very high. so for good conductors,

$$J \gg \frac{\partial \bar{D}}{\partial t} \text{ and } \epsilon_r = 0$$

point form :-

$$\nabla \cdot \bar{D} = 0$$

$$\nabla \times \bar{E} = -\frac{\partial \bar{B}}{\partial t}$$

$$\nabla \times \bar{H} = \bar{J}$$

$$\nabla \cdot \bar{B} = 0$$

Integral form :-

$$\oint_S \bar{D} \cdot d\bar{s} = 0$$

$$\oint \bar{E} \cdot d\bar{L} = - \int_S \frac{\partial \bar{B}}{\partial t} \cdot d\bar{s}$$

$$\oint \bar{H} \cdot d\bar{L} = I = \int_S \bar{J} \cdot d\bar{s}$$

$$\oint_S \bar{B} \cdot d\bar{s} = 0$$

Maxwell's equations :-

Differential form	Integral form	Significance
$\nabla \cdot \bar{D} = \rho_v$	$\oint_S \bar{D} \cdot d\bar{s} = \int_V \rho_v dV$	Gauss's Law (Electric)
$\nabla \times \bar{E} = -\frac{\partial \bar{B}}{\partial t}$	$\oint \bar{E} \cdot d\bar{L} = - \int_S \frac{\partial \bar{B}}{\partial t} \cdot d\bar{s}$	Faraday's Law
$\nabla \times \bar{H} = \bar{J} + \frac{\partial \bar{D}}{\partial t}$	$\oint \bar{H} \cdot d\bar{L} = I + \int_S \frac{\partial \bar{D}}{\partial t} \cdot d\bar{s}$	Ampere's Circuital Law
$\nabla \cdot \bar{B} = 0$	$\oint \bar{B} \cdot d\bar{s} = 0$	Gauss's Law (Magnetic)

conditions:

Boundary conditions for electric field:

consider the boundary surface b/w two media - medium 1 of relative permittivity ϵ_{r1} and medium 2 of relative permittivity ϵ_{r2} .

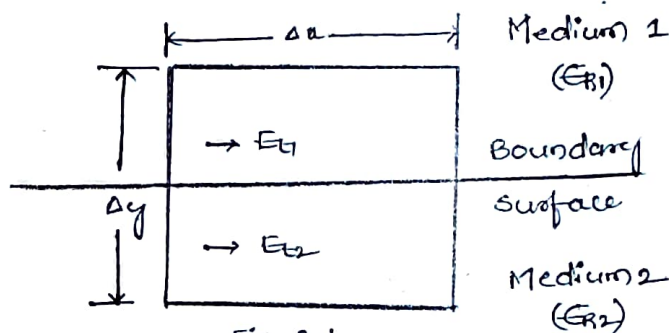


Fig. 3.1

Let the tangential components of the electric field intensity at the boundary surface be E_1 and E_2 respectively in two media.

consider a small closed path (a rectangular) with dimensions Δx and Δy .

Applying Maxwell's eqn derived from Faraday's Law

$$\oint \mathbf{E} \cdot d\mathbf{L} = - \int \frac{\partial \mathbf{B}}{\partial t} \cdot d\mathbf{s}$$

the necessary boundary relation can be obtained.

It is seen that as $\Delta y \rightarrow 0$, the area of the closed path also approaches zero, and hence the R.H.S becomes zero even if the magnetic flux density $\mathbf{B}$ has a finite value.

$$\therefore \oint \mathbf{E} \cdot d\mathbf{L} = 0$$

$$E_1 \Delta x - E_2 \Delta x = 0$$

$$E_1 \Delta x = E_2 \Delta x$$

$$\boxed{E_1 = E_2}$$

The tangential components of E in both media are equal.

Note: If one of the media, say medium-2 is a perfect conductor, we have $E_2 = 0$, since the electric field inside a perfect conductor is zero.

$$E_{t1} = E_{t2} = 0$$

Let D_{1N} and D_{2N} denote the normal components.

consider a small cylindrical gaussian surface cutting into the boundary.

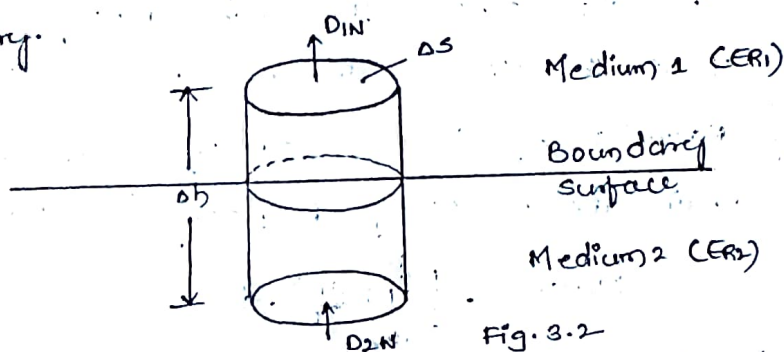


Fig. 3.2

The necessary boundary relation can be obtained by applying the Maxwell's equation derived from Gauss' Law for electric circuit

$$\oint_S \mathbf{D} \cdot d\mathbf{s} = \int_{vol} \rho_v dv$$

As $2h \rightarrow 0$, the cylinder progressively shrinks in volume.

We have $D_{1N} \Delta S - D_{2N} \Delta S = \rho_s \Delta S$ where ρ_s = surface charge density, and generally $\rho_s = 0$, since usually there is no charge on the boundary surface, if both media are dielectrics.

$$D_{1N} \Delta S - D_{2N} \Delta S = 0$$

$$D_{1N} = D_{2N}$$

Thus the normal component of D is continuous.

Note: If one of the two media - say medium-2 is a conductor then $D_{2N} = 0$, but ρ_s is not zero, as there can be charge on a conductor surface.

$$\therefore \text{We get } D_{1N} \Delta S - 0 = \rho_s \Delta S$$

$$D_{1N} = \rho_s$$

Boundary conditions for Magnetic fields:

consider the interface b/w two magnetic materials of relative permeabilities μ_{r1} and μ_{r2} as shown.

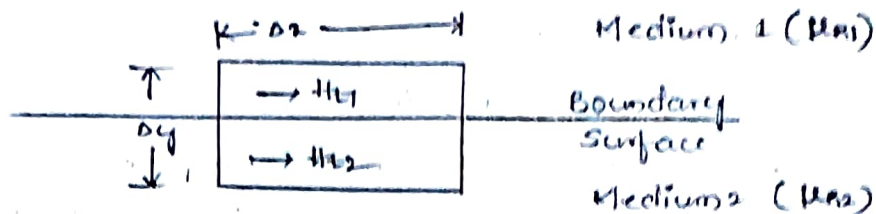


Fig. 3.3

Let H_1 and H_2 denote the tangential components of the magnetic field intensity H in the two media.

We apply Maxwell's equation derived from Ampere's circuital law to obtain the necessary boundary condition.

$$\oint H \cdot dL = \int_S \left(J + \frac{\partial D}{\partial t} \right) \cdot dS$$

It is seen that as $\Delta y \rightarrow 0$, $\frac{\partial D}{\partial t} \rightarrow 0$ i.e. the displacement current density becomes zero.

The conduction current density J is also normally zero at a dielectric interface.

Let J be assumed as zero.

$$H_1 \Delta x - H_2 \Delta x = 0$$

$$H_1 = H_2$$

The tangential components of H are equal.

If one of the media - say medium-2 is a perfect conductor

$$H_2 = 0.$$

$\therefore H_1 = J_s$ where $J_s \rightarrow$ Surface current density.

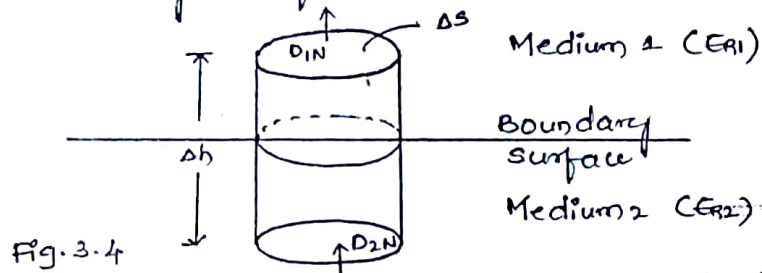
NOTE: If there is a current sheet of current density J_s at the boundary surface of two dielectric media, then

$$H_1 - H_2 = J_s$$

Let us apply Maxwell's equation derived from Gauss's Law for magnetic fields

$$\oint_S \mathbf{B} \cdot d\mathbf{s} = 0$$

consider a small cylindrical gaussian surface which cuts into the boundary surface



It can be seen that as $\Delta h \rightarrow 0$, the cylindrical surface of the pill box considered becomes zero.

$$B_{1N} \Delta s - B_{2N} \Delta s = 0$$

$$B_{1N} = B_{2N}$$

Hence the normal component of $\mathbf{B}$ is continuous.

Summary of Maxwell's equations:

	Point differential form	Integral form
Static fields	$\nabla \cdot \mathbf{D} = \rho_v$	$\oint_S \mathbf{D} \cdot d\mathbf{s} = \int_{Vol} \rho_v dv$
	$\nabla \times \mathbf{E} = 0$	$\oint \mathbf{E} \cdot d\mathbf{L} = 0$
	$\nabla \times \mathbf{H} = \mathbf{J}$	$\oint \mathbf{H} \cdot d\mathbf{L} = \int_S \mathbf{J} \cdot d\mathbf{s}$
	$\nabla \cdot \mathbf{B} = 0$	$\oint_S \mathbf{B} \cdot d\mathbf{s} = 0$
Time varying fields.	$\nabla \cdot \mathbf{D} = \rho_v$	$\oint_S \mathbf{D} \cdot d\mathbf{s} = \int_{Vol} \rho_v dv$
	$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}$	$\oint \mathbf{E} \cdot d\mathbf{L} = -\int_S \frac{\partial \mathbf{B}}{\partial t} \cdot d\mathbf{s}$
	$\nabla \times \mathbf{H} = \mathbf{J} + \frac{\partial \mathbf{D}}{\partial t}$	$\oint \mathbf{H} \cdot d\mathbf{L} = \int_S \left(\mathbf{J} + \frac{\partial \mathbf{D}}{\partial t} \right) \cdot d\mathbf{s}$
	$\nabla \cdot \mathbf{B} = 0$	$\oint_S \mathbf{B} \cdot d\mathbf{s} = 0$

Faraday's Law:

$$e = -\frac{d\phi}{dt} = \oint E \cdot dL$$

expressions for induced e.m.f

$$\oint E \cdot dL = - \int_s \frac{\partial B}{\partial t} \cdot ds$$

$$\nabla \times E = - \frac{\partial B}{\partial t}$$

conduction current density $J_c = \sigma E$

Displacement current density $J_D = \frac{\partial D}{\partial t}$

Maxwell's equation ÷

General form

Point form	Integral form
$\nabla \cdot D = \rho_v$	$\oint_s D \cdot ds = \int_v \rho_v dv$
$\nabla \times E = - \frac{\partial B}{\partial t}$	$\oint E \cdot dL = - \int_s \frac{\partial B}{\partial t} \cdot ds$
$\nabla \times H = J + \frac{\partial D}{\partial t}$	$\oint H \cdot dL = I + \int_s \frac{\partial D}{\partial t} \cdot ds$
$\nabla \cdot B = 0$	$\oint_s B \cdot ds = 0$

For free space ÷

Point form	Integral form
$\nabla \cdot D = 0$	$\oint_s D \cdot ds = 0$
$\nabla \times E = - \frac{\partial B}{\partial t}$	$\oint E \cdot dL = - \int_s \frac{\partial B}{\partial t} \cdot ds$
$\nabla \times H = \frac{\partial D}{\partial t}$	$\oint H \cdot dL = \int_s \frac{\partial D}{\partial t} \cdot ds$
$\nabla \cdot B = 0$	$\oint_s B \cdot ds = 0$

For good conductors :

Point Form	Integral Form
$\nabla \cdot \mathbf{D} = 0$	$\oint_S \mathbf{D} \cdot d\mathbf{s} = 0$
$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}$	$\oint \mathbf{E} \cdot d\mathbf{L} = - \int_S \frac{\partial \mathbf{B}}{\partial t} \cdot d\mathbf{s}$
$\nabla \times \mathbf{H} = \mathbf{J}$	$\oint \mathbf{H} \cdot d\mathbf{L} = \int_S \mathbf{J} \cdot d\mathbf{s}$
$\nabla \cdot \mathbf{B} = 0$	$\oint_S \mathbf{B} \cdot d\mathbf{s} = 0$

For harmonically varying fields :

Point Form	Integral Form
$\nabla \cdot \mathbf{D} = \rho_v$	$\oint_S \mathbf{D} \cdot d\mathbf{s} = \int_V \rho_v dv$
$\nabla \times \mathbf{E} = -j\omega\mu\mathbf{H}$	$\oint \mathbf{E} \cdot d\mathbf{L} = -j\omega\mu \int_S \mathbf{H} \cdot d\mathbf{s}$
$\nabla \times \mathbf{H} = (\sigma + j\omega\epsilon)\mathbf{E}$	$\oint \mathbf{H} \cdot d\mathbf{L} = (\sigma + j\omega\epsilon) \int_S \mathbf{E} \cdot d\mathbf{s}$
$\nabla \cdot \mathbf{B} = 0$	$\oint_S \mathbf{B} \cdot d\mathbf{s} = 0$

INTRODUCTION

In Part 2 (Chapters 4–6) of this text, we mainly concentrated our efforts on electrostatic fields denoted by $\mathbf{E}(x, y, z)$; Part 3 (Chapters 7 and 8) was devoted to magnetostatic fields represented by $\mathbf{H}(x, y, z)$. We have therefore restricted our discussions to static, or time-invariant, EM fields. Henceforth, we shall examine situations in which electric and magnetic fields are dynamic, or time varying. It should be mentioned first that in static EM fields, electric and magnetic fields are independent of each other, whereas in dynamic EM fields, the two fields are interdependent. In other words, a time-varying electric field necessarily involves a corresponding time-varying magnetic field. Second, time-varying EM fields, represented by $\mathbf{E}(x, y, z, t)$ and $\mathbf{H}(x, y, z, t)$, are of more practical value than static EM fields. However, familiarity with static fields provides a good background for understanding dynamic fields. Third, recall that electrostatic fields are usually produced by static electric charges, whereas magnetostatic fields are due to motion of electric charges with uniform velocity (direct current) or static magnetic charges (magnetic poles); time-varying fields or waves are usually due to accelerated charges or time-varying currents such as shown in Figure 9.1. Any pulsating current will produce radiation (time-varying fields). It is worth noting that pulsating current of the type shown in Figure 9.1(b) is the cause of radiated emission in digital logic boards. In summary:

stationary charges $\rightarrow$ electrostatic fields
steady currents $\rightarrow$ magnetostatic fields
time-varying currents $\rightarrow$ electromagnetic fields (or waves)

Our aim in this chapter is to lay a firm foundation for our subsequent studies. This will involve introducing two major concepts: (1) electromotive force based on Faraday's experiments and (2) displacement current, which resulted from Maxwell's hypothesis. As a result of these concepts, Maxwell's equations as presented in Section 7.6 and the boundary conditions for static EM fields

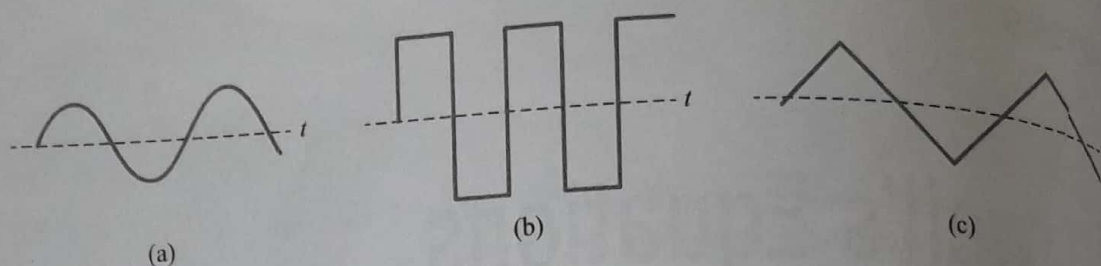


Figure 9.1 Examples of time-varying current: (a) sinusoidal, (b) rectangular, (c) triangular.

will be modified to account for the time variation of the fields. Maxwell's equations, which summarize the laws of electromagnetism, shall be the basis of our discussions in the remaining part of the text. For this reason, Section 9.5 should be regarded as the heart of this text.

9.2 FARADAY'S LAW

After Oersted's experimental discovery (upon which Biot–Savart and Ampère based their laws) that a steady current produces a magnetic field, it seemed logical to find out whether magnetism would produce electricity. In 1831, about 11 years after Oersted's discovery, Michael Faraday in London and Joseph Henry in New York discovered that a time-varying magnetic field would produce an electric current.¹

According to Faraday's experiments, a static magnetic field produces no current flow; but in a closed circuit, a time-varying field produces an induced voltage (called *electromotive force* or simply *emf*) that causes a flow of current.

Faraday discovered that the **induced emf**, V_{emf} (in volts), in any closed circuit is equal to the time rate of change of the magnetic flux linkage by the circuit.

This is called *Faraday's law*, and it can be expressed as

$$V_{\text{emf}} = -\frac{d\lambda}{dt} = -N \frac{d\Psi}{dt} \quad (9.1)$$

where $\lambda = N\Psi$ is the flux linkage, N is the number of turns in the circuit, and Ψ is the flux through each turn. The negative sign shows that the induced voltage acts in such a way as to oppose the flux producing it. This behavior is described by *Lenz's law*,² which emphasizes that the direction of current flow in the circuit is such that the induced magnetic field produced by the induced current will oppose the change in original magnetic field.

Recall that we described an electric field as one in which electric charges experience force. The electric fields considered so far are caused by electric charges; in such fields, the flux lines begin and end on the charges. However, electric fields of other kinds are not directly caused by electric charges. These are emf-produced fields. Sources of emf include electric generators, batteries, thermocouples, fuel cells, and photovoltaic cells, which all convert nonelectrical energy into electrical energy.

¹For details on the experiments of Michael Faraday (1791–1867) and Joseph Henry (1797–1878), see W. F. Magie, *A Source Book in Physics*. Cambridge, MA: Harvard Univ. Press, 1963, pp. 472–519.

²After Heinrich Friedrich Emil Lenz (1804–1865), a Russian professor of physics.

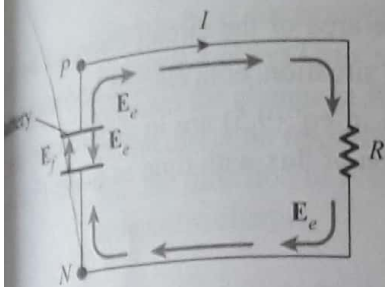


Figure 9.2 A circuit showing emf-producing field $\mathbf{E}_f$ and electrostatic field $\mathbf{E}_e$.

Consider the electric circuit of Figure 9.2, where the battery is a source of emf. The electrochemical action of the battery results in an emf-produced field $\mathbf{E}_f$. Due to the accumulation of charge at the battery terminals, an electrostatic field $\mathbf{E}_e (= -\nabla V)$ also exists. The total electric field at any point is

$$\mathbf{E} = \mathbf{E}_f + \mathbf{E}_e \quad (9.2)$$

Note that $\mathbf{E}_f$ is zero outside the battery, $\mathbf{E}_f$ and $\mathbf{E}_e$ have opposite directions in the battery, and the direction of $\mathbf{E}_e$ inside the battery is opposite to that outside it. If we integrate eq. (9.2) over the closed circuit, we have

$$\oint_L \mathbf{E} \cdot d\mathbf{l} = \oint_L \mathbf{E}_f \cdot d\mathbf{l} + 0 = \int_N^P \mathbf{E}_f \cdot d\mathbf{l} \quad (\text{through battery}) \quad (9.3a)$$

where $\oint_L \mathbf{E}_e \cdot d\mathbf{l} = 0$ because $\mathbf{E}_e$ is conservative. The emf of the battery is the line integral of the emf-produced field, that is,

$$V_{\text{emf}} = \int_N^P \mathbf{E}_f \cdot d\mathbf{l} = -\int_N^P \mathbf{E}_e \cdot d\mathbf{l} = IR \quad (9.3b)$$

since $\mathbf{E}_f$ and $\mathbf{E}_e$ are equal but opposite within the battery (see Figure 9.2). It may also be regarded as the potential difference ($V_P - V_N$) between the battery's open-circuit terminals. It is important to note the following facts.

1. An electrostatic field $\mathbf{E}_e$ cannot maintain a steady current in a closed circuit, since $\oint_L \mathbf{E}_e \cdot d\mathbf{l} = 0 = IR$.
2. An emf-produced field $\mathbf{E}_f$ is nonconservative.
3. Except in electrostatics, voltage and potential difference are usually not equivalent.

It should be noted that when an electron moves from P to N through the battery, against the direction of the electrostatic field ($\mathbf{E}_e$), the work has to be done by the battery and the corresponding emf $\mathbf{E}_f$. The (potential) energy thus gained is dissipated in the resistor. $\mathbf{E}_f$ being a nonconservative field enables the conversion of the chemical energy in the battery to the heat loss in the resistor. On the other hand, $\mathbf{E}_e$ is a conservative field and cannot do energy conversion or do any useful work.

9.3 TRANSFORMER AND MOTIONAL ELECTROMOTIVE FORCES

Having considered the connection between emf and electric field, we may examine how Faraday's law links electric and magnetic fields. For a circuit with a single turn ($N = 1$), eq. (9.1) becomes

$$V_{\text{emf}} = -\frac{d\psi}{dt} \quad (9.4)$$

In terms of $\mathbf{E}$ and $\mathbf{B}$, eq. (9.4) can be written as

$$V_{\text{emf}} = \oint_L \mathbf{E} \cdot d\mathbf{l} = -\frac{d}{dt} \int_S \mathbf{B} \cdot d\mathbf{S} \quad (9.5)$$

where ψ has been replaced by $\int_S \mathbf{B} \cdot d\mathbf{S}$ and S is the surface area of the circuit bounded by the closed path L . It is clear from eq. (9.5) that in a time-varying situation, both electric and magnetic fields are present and are interrelated. Note that $d\mathbf{l}$ and $d\mathbf{S}$ in eq. (9.5) are in accordance with the right-hand rule as well as Stokes's theorem. The variation of flux with time as in eq. (9.1) or eq. (9.5) may be caused in three ways:

1. By having a stationary loop in a time-varying $\mathbf{B}$ field
2. By having a time-varying loop area in a static $\mathbf{B}$ field
3. By having a time-varying loop area in a time-varying $\mathbf{B}$ field

Each of these will be considered separately.

A. Stationary Loop in Time-Varying $\mathbf{B}$ Field (Transformer emf)

In Figure 9.3 a stationary conducting loop is in a time-varying magnetic $\mathbf{B}$ field. As integration over space and differentiation with time are independent operations,

$$-\frac{d}{dt} \int_S \mathbf{B} \cdot d\mathbf{S} = - \int_S \frac{d}{dt} (\mathbf{B} \cdot d\mathbf{S}) \quad (9.6a)$$

Equation (9.5) becomes

$$V_{\text{emf}} = \oint_L \mathbf{E} \cdot d\mathbf{l} = - \int_S \frac{\partial \mathbf{B}}{\partial t} \cdot d\mathbf{S} \quad (9.6b)$$

This emf induced by the time-varying current (producing the time-varying $\mathbf{B}$ field) in a stationary loop is often referred to as *transformer emf* in power analysis, since it is due to transformer action. By applying Stokes's theorem to the middle term in eq. (9.6), we obtain

$$\int_S (\nabla \times \mathbf{E}) \cdot d\mathbf{S} = - \int_S \frac{\partial \mathbf{B}}{\partial t} \cdot d\mathbf{S} \quad (9.7)$$

For the two integrals to be equal, their integrands must be equal; that is,

$$\nabla \times \mathbf{E} = - \frac{\partial \mathbf{B}}{\partial t} \quad (9.8)$$

This is one of the Maxwell's equations for time-varying fields. It shows that the time-varying $\mathbf{E}$ field is not conservative ($\nabla \times \mathbf{E} \neq \mathbf{0}$). This does not imply that the principles of energy conservation are violated. The work done in taking a charge about a closed path in a time-varying

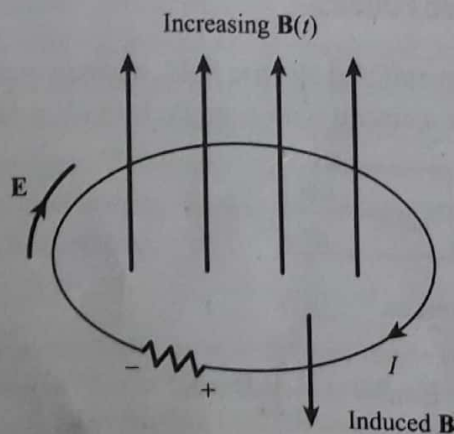


Figure 9.3 Induced emf due to a stationary loop in a time-varying $\mathbf{B}$ field.

electric field, for example, is due to the energy from the time-varying magnetic field. Observe that Figure 9.3 obeys Lenz's law: the induced current I flows such as to produce a magnetic field that opposes the change in $\mathbf{B}(t)$. Also, I flows in the direction opposite to that of electrons which move in the direction opposite to that of $\mathbf{E}$ (that is I is along the direction of $\mathbf{E}$). It should be recalled that the direction of $\mathbf{E}$ is from a higher potential to a lower potential or from a positive charge to a negative charge.

B. Moving Loop in Static B Field (Motional emf)

When a conducting loop is moving in a static $\mathbf{B}$ field, an emf is induced in the loop. We recall from eq. (8.2) that the force on a charge moving with uniform velocity $\mathbf{u}$ in a magnetic field $\mathbf{B}$ is

$$\mathbf{F}_m = Q\mathbf{u} \times \mathbf{B} \quad (8.2)$$

We define the *motional electric field* $\mathbf{E}_m$ as

$$\mathbf{E}_m = \frac{\mathbf{F}_m}{Q} = \mathbf{u} \times \mathbf{B} \quad (9.9)$$

If we consider a conducting loop, moving with uniform velocity $\mathbf{u}$ as consisting of a large number of free electrons, the emf induced in the loop is

$$V_{\text{emf}} = \oint_L \mathbf{E}_m \cdot d\mathbf{l} = \oint_L (\mathbf{u} \times \mathbf{B}) \cdot d\mathbf{l} \quad (9.10)$$

This type of emf is called *motional emf* or *flux-cutting emf* because it is due to motional action. It is the kind of emf found in electrical machines such as motors, generators, and alternators. Figure 9.4 illustrates a two-pole dc machine with one armature coil and a two-bar commutator. Although the analysis of the dc machine is beyond the scope of this text, we can see that voltage is generated as the coil rotates within the magnetic field. Another example of motional emf is illustrated in Figure 9.5, where a rod is moving between a pair of rails. In this example, $\mathbf{B}$ and $\mathbf{u}$ are perpendicular, so eq. (9.9) in conjunction with eq. (8.2) becomes

$$\mathbf{F}_m = I\boldsymbol{\ell} \times \mathbf{B} \quad (9.11)$$

or

$$F_m = I\ell B \quad (9.12)$$

and eq. (9.10) becomes

$$V_{\text{emf}} = uB\ell \quad (9.13)$$

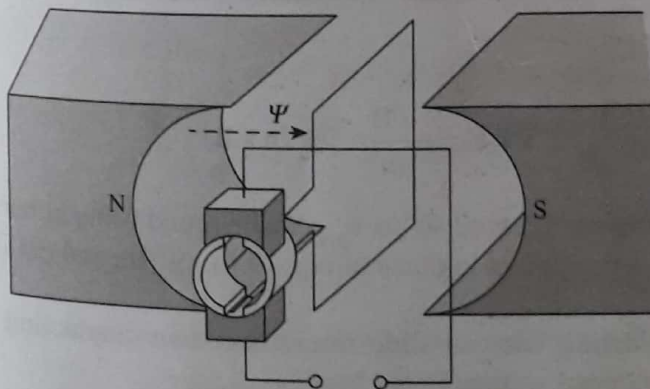


Figure 9.4 A direct-current machine.

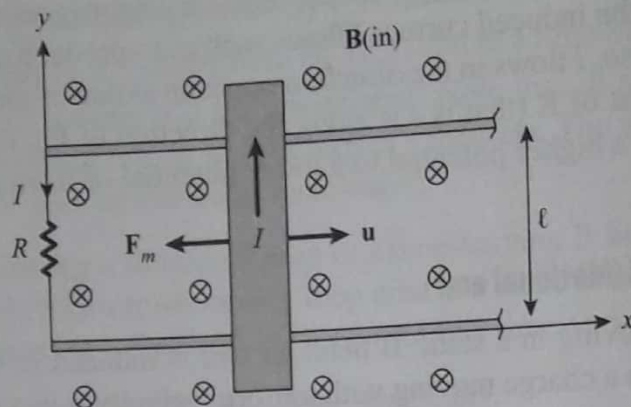


Figure 9.5 Induced emf due to a moving loop in a static $\mathbf{B}$ field.

By applying Stokes's theorem to eq. (9.10), we have

$$\int_S (\nabla \times \mathbf{E}_m) \cdot d\mathbf{S} = \int_S \nabla \times (\mathbf{u} \times \mathbf{B}) \cdot d\mathbf{S}$$

or

$$\boxed{\nabla \times \mathbf{E}_m = \nabla \times (\mathbf{u} \times \mathbf{B})} \quad (9.14)$$

Notice that unlike eq. (9.6), there is no need for a minus sign in eq. (9.10) because Lenz's law is already accounted for.

To apply eq. (9.10) is not always easy; some care must be exercised. The following points should be noted.

1. The integral in eq. (9.10) is zero along the portion of the loop where $\mathbf{u} = \mathbf{0}$. Thus $d\mathbf{l}$ is taken along the portion of the loop that is cutting the field (along the rod in Figure 9.5), where $\mathbf{u}$ has nonzero value.
2. The direction of the induced current is the same as that of $\mathbf{E}_m$ or $\mathbf{u} \times \mathbf{B}$. The limits of the integral in eq. (9.10) are selected in the direction opposite to the induced current, thereby satisfying Lenz's law. In Figure 9.5, for example, the integration over L is along $-\mathbf{a}_y$, whereas induced current flows in the rod along $\mathbf{a}_y$.

C. Moving Loop in Time-Varying Field

In the general case, a moving conducting loop is in a time-varying magnetic field. Both transformer emf and motional emf are present. Combining eqs. (9.6) and (9.10) gives the total emf as

$$\boxed{V_{\text{emf}} = \oint_L \mathbf{E} \cdot d\mathbf{l} = - \int_S \frac{\partial \mathbf{B}}{\partial t} \cdot d\mathbf{S} + \oint_L (\mathbf{u} \times \mathbf{B}) \cdot d\mathbf{l}} \quad (9.15)$$

or from eqs. (9.8) and (9.14),

$$\nabla \times \mathbf{E} = - \frac{\partial \mathbf{B}}{\partial t} + \nabla \times (\mathbf{u} \times \mathbf{B}) \quad (9.16)$$

Note that eq. (9.15) is equivalent to eq. (9.4), so V_{emf} can be found using either eq. (9.15) or (9.4). In fact, eq. (9.4) can always be applied in place of eqs. (9.6), (9.10), and (9.15).

9.4 DISPLACEMENT CURRENT

In Section 9.3 we have essentially reconsidered Maxwell's curl equation for electrostatic fields and modified it for time-varying situations to satisfy Faraday's law. We shall now reconsider Maxwell's curl equation for magnetic fields (Ampère's circuit law) for time-varying conditions.

For static EM fields, we recall that

$$\nabla \times \mathbf{H} = \mathbf{J} \quad (9.17)$$

But the divergence of the curl of any vector field is identically zero (see Example 3.10). Hence,

$$\nabla \cdot (\nabla \times \mathbf{H}) = 0 = \nabla \cdot \mathbf{J} \quad (9.18)$$

The continuity of current in eq. (5.43), however, requires that

$$\nabla \cdot \mathbf{J} = -\frac{\partial \rho_v}{\partial t} \neq 0 \quad (9.19)$$

Thus eqs. (9.18) and (9.19) are obviously incompatible for time-varying conditions. We must modify eq. (9.17) to agree with eq. (9.19). To do this, we add a term to eq. (9.17) so that it becomes

$$\nabla \times \mathbf{H} = \mathbf{J} + \mathbf{J}_d \quad (9.20)$$

where $\mathbf{J}_d$ is to be determined and defined. Again, the divergence of the curl of any vector is zero. Hence:

$$\nabla \cdot (\nabla \times \mathbf{H}) = 0 = \nabla \cdot \mathbf{J} + \nabla \cdot \mathbf{J}_d \quad (9.21)$$

In order for eq. (9.21) to agree with eq. (9.19),

$$\nabla \cdot \mathbf{J}_d = -\nabla \cdot \mathbf{J} = \frac{\partial \rho_v}{\partial t} = \frac{\partial}{\partial t} (\nabla \cdot \mathbf{D}) = \nabla \cdot \frac{\partial \mathbf{D}}{\partial t} \quad (9.22a)$$

or

$$\boxed{\mathbf{J}_d = \frac{\partial \mathbf{D}}{\partial t}} \quad (9.22b)$$

Substituting eq. (9.22b) into eq. (9.20) results in

$$\boxed{\nabla \times \mathbf{H} = \mathbf{J} + \frac{\partial \mathbf{D}}{\partial t}} \quad (9.23)$$

This is Maxwell's equation (based on Ampère's circuit law) for a time-varying field. The term $\mathbf{J}_d = \partial \mathbf{D} / \partial t$ is known as *displacement current density* and $\mathbf{J}$ is the conduction current density ($\mathbf{J} = \sigma \mathbf{E}$).³ The insertion of $\mathbf{J}_d$ into eq. (9.17) was one of the major contributions of Maxwell. Without the term $\mathbf{J}_d$, the propagation of electromagnetic waves (e.g., radio or TV waves) would be impossible. At low frequencies, $\mathbf{J}_d$ is usually neglected compared with $\mathbf{J}$. However, at radio frequencies, the two terms are comparable. At the time of Maxwell, high-frequency sources were not available and eq. (9.23) could not be verified experimentally. It was years later that Hertz succeeded in generating and detecting radio waves, thereby verifying eq. (9.23). This is one of the rare cases of a mathematical argument paving the way for experimental investigation.

Based on the displacement current density, we define the *displacement current* as

$$I_d = \int_S \mathbf{J}_d \cdot d\mathbf{S} = \int_S \frac{\partial \mathbf{D}}{\partial t} \cdot d\mathbf{S} \quad (9.24)$$

We must bear in mind that displacement current is a result of time-varying electric field. A typical example of such current is the current through a capacitor when an alternating voltage source is applied to its plates. This example, shown in Figure 9.10, serves to illustrate the need for the displacement current. Applying an unmodified form of Ampère's circuit law to a closed path L shown in Figure 9.10(a) gives

$$\oint_L \mathbf{H} \cdot d\mathbf{l} = \int_{S_1} \mathbf{J} \cdot d\mathbf{S} = I_{\text{enc}} = I \quad (9.25)$$

where I is the current through the conductor and S_1 is the flat surface bounded by L . If we use the balloon-shaped surface S_2 that passes between the capacitor plates, as in Figure 9.10(b),

$$\oint_L \mathbf{H} \cdot d\mathbf{l} = \int_{S_2} \mathbf{J} \cdot d\mathbf{S} = I_{\text{enc}} = 0 \quad (9.26)$$

because no conduction current ($\mathbf{J} = 0$) flows through S_2 . This is contradictory in view of the fact that the same closed path L is used. To resolve the conflict, we need to include the displacement current in Ampère's circuit law. The total current density is $\mathbf{J} + \mathbf{J}_d$. In eq. (9.25), $\mathbf{J}_d = 0$, so that the equation remains valid. In eq. (9.26), $\mathbf{J} = 0$, so that

³Recall that we also have $\mathbf{J} = \rho_v \mathbf{u}$ as the convection current density.

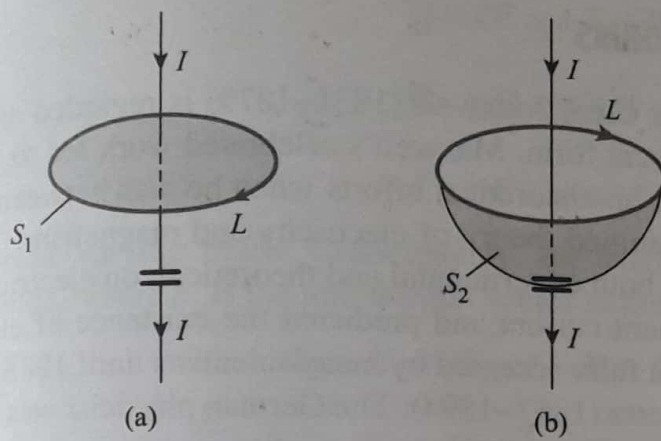


Figure 9.10 Two surfaces of integration showing the need for $\mathbf{J}_d$ in Ampère's circuit law.

$$\oint_L \mathbf{H} \cdot d\mathbf{l} = \int_{S_2} \mathbf{J}_d \cdot d\mathbf{S} = \frac{d}{dt} \int_{S_2} \mathbf{D} \cdot d\mathbf{S} = \frac{dQ}{dt} = I \quad (9.27)$$

So we obtain the same current for either surface, although it is conduction current in S_1 and displacement current in S_2 .

9.5 MAXWELL'S EQUATIONS IN FINAL FORMS

The Scottish physicist James Clerk Maxwell (1831–1879) is regarded as the founder of electromagnetic theory in its present form. Maxwell's celebrated work led to the discovery of electromagnetic waves.⁴ Through his theoretical efforts when he was between 35 and 40 years old, Maxwell published the first unified theory of electricity and magnetism. The theory comprised all previously known results, both experimental and theoretical, on electricity and magnetism. It further introduced displacement current and predicted the existence of electromagnetic waves. Maxwell's equations were not fully accepted by many scientists until 1888, when they were confirmed by Heinrich Rudolf Hertz (1857–1894). The German physicist was successful in generating and detecting radio waves.

The laws of electromagnetism that Maxwell put together in the form of four equations were presented in Table 7.2 in Section 7.6 for static conditions. The more generalized forms of these equations are those for time-varying conditions shown in Table 9.1. We notice from the table that the divergence equations remain the same, while the curl equations have been modified. The integral form of Maxwell's equations depicts the underlying physical laws, whereas the differential form is used more frequently in solving problems. For a field to "qualify" as an electromagnetic field, it must satisfy all four Maxwell's equations. The importance of Maxwell's equations cannot be overemphasized because they summarize all known laws of electromagnetism. We shall often refer to them in the remainder of this text.

Since this section is meant to be a compendium of our discussion in this text, it is worthwhile to mention other equations that go hand in hand with Maxwell's equations. The Lorentz force equation

$$\mathbf{F} = Q(\mathbf{E} + \mathbf{u} \times \mathbf{B}) \quad (9.28)$$

is associated with Maxwell's equations. Also the equation of continuity

$$\nabla \cdot \mathbf{J} = -\frac{\partial \rho_v}{\partial t} \quad (9.29)$$

is implicit in Maxwell's equations. The concepts of linearity, isotropy, and homogeneity of a material medium still apply for time-varying fields; in a linear, homogeneous, and isotropic medium characterized by σ , ϵ , and μ , the constitutive relations

Table 9.1 Generalized Forms of Maxwell's Equations

Differential Form	Integral Form	Remarks
$\nabla \cdot \mathbf{D} = \rho_v$	$\oint_S \mathbf{D} \cdot d\mathbf{S} = \int_V \rho_v dV$	Gauss's law
$\nabla \cdot \mathbf{B} = 0$	$\oint_S \mathbf{B} \cdot d\mathbf{S} = 0$	Nonexistence of isolated magnetic charge*
$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}$	$\oint_L \mathbf{E} \cdot d\mathbf{l} = -\frac{\partial}{\partial t} \int_S \mathbf{B} \cdot d\mathbf{S}$	Faraday's law
$\nabla \times \mathbf{H} = \mathbf{J} + \frac{\partial \mathbf{D}}{\partial t}$	$\oint_L \mathbf{H} \cdot d\mathbf{l} = \int_S \left(\mathbf{J} + \frac{\partial \mathbf{D}}{\partial t} \right) \cdot d\mathbf{S}$	Ampère's circuit law

*This is also referred to as Gauss's law for magnetic fields.

⁴Maxwell's work can be found in his two-volume *Treatise on Electricity and Magnetism* (New York: Dover, 1954).

$$\mathbf{D} = \epsilon \mathbf{E} = \epsilon_0 \mathbf{E} + \mathbf{P} \quad (9.30a)$$

$$\mathbf{B} = \mu \mathbf{H} = \mu_0 (\mathbf{H} + \mathbf{M}) \quad (9.30b)$$

$$\mathbf{J} = \sigma \mathbf{E} + \rho_v \mathbf{u} \quad (9.30c)$$

hold for time-varying fields. Consequently, the boundary conditions remain valid for time-varying fields, where $\mathbf{a}_n$ is the unit normal vector to the boundary.

$$E_{1t} - E_{2t} = 0 \quad \text{or} \quad (\mathbf{E}_1 - \mathbf{E}_2) \times \mathbf{a}_n = 0 \quad (9.31a)$$

$$H_{1t} - H_{2t} = K \quad \text{or} \quad (\mathbf{H}_1 - \mathbf{H}_2) \times \mathbf{a}_n = \mathbf{K} \quad (9.31b)$$

$$D_{1n} - D_{2n} = \rho_s \quad \text{or} \quad (\mathbf{D}_1 - \mathbf{D}_2) \cdot \mathbf{a}_n = \rho_s \quad (9.31c)$$

$$B_{1n} - B_{2n} = 0 \quad \text{or} \quad (\mathbf{B}_2 - \mathbf{B}_1) \cdot \mathbf{a}_n = 0 \quad (9.31d)$$

However, for a perfect conductor ($\sigma \approx \infty$), in a time-varying field,

$$\mathbf{E} = 0, \quad \mathbf{H} = 0, \quad \mathbf{J} = 0 \quad (9.32)$$

and hence,

$$\mathbf{B}_n = 0, \quad \mathbf{E}_t = 0 \quad (9.33)$$

For a perfect dielectric ($\sigma \approx 0$), eqs. (9.31) hold except that $\mathbf{K} = 0$. Though eqs. (9.28) to (9.33) are not Maxwell's equations, they are associated with them.

To complete this summary section, we present a structure linking the various potentials and vector fields of the electric and magnetic fields in Figure 9.11. This electromagnetic flow diagram helps with the visualization of the basic relationships between field quantities. It also shows that it is usually possible to find alternative formulations, for a given problem, in a relatively simple manner. It should be noted that in Figure 9.11(b) and (c), we introduce ρ^m as the free magnetic density (similar to ρ_v), which is, of course, zero, $\mathbf{A}_e$ as the electric vector potential (analogous to $\mathbf{A}$), and $\mathbf{J}_m$ as the magnetic current density (analogous to $\mathbf{J}$). Using terms from stress analysis, the principal relationships are typified as follows:

(a) compatibility equations

$$\nabla \cdot \mathbf{B} = \rho^m = 0 \quad (9.34)$$

and

$$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t} = \mathbf{J}_m \quad (9.35)$$

(b) constitutive equations

$$\mathbf{B} = \mu \mathbf{H} \quad (9.36)$$

and

$$\mathbf{D} = \epsilon \mathbf{E} \quad (9.37)$$

(c) equilibrium equations

$$\nabla \cdot \mathbf{D} = \rho_v \quad (9.38)$$

and

Electromagnetic Wave Propagation

A wise man rules his passions, a fool obeys them.

—PUBLILIUS SYRUS

10.1 INTRODUCTION

Our first application of Maxwell's equations will be in relation to electromagnetic wave propagation. The existence of EM waves, predicted by Maxwell's equations, was first investigated by Heinrich Hertz. After several calculations and experiments, Hertz succeeded in generating and detecting radio waves, which are sometimes called Hertzian waves in his honor.

In general, **waves** are means of transporting energy or information.

Typical examples of EM waves include radio waves, TV signals, radar beams, and light rays. All forms of EM energy share three fundamental characteristics: they all travel at high velocity; in traveling, they assume the properties of waves; and they radiate outward from a source. The problem of radiation will be addressed in Chapter 13.

In this chapter, our major goal is to solve Maxwell's equations and describe EM wave motion in the following media:

1. Free space ($\sigma = 0, \epsilon = \epsilon_0, \mu = \mu_0$)
2. Lossless dielectrics ($\sigma \approx 0, \epsilon = \epsilon_r \epsilon_0, \mu = \mu_r \mu_0$, or $\sigma \ll \omega \epsilon$)
3. Lossy dielectrics ($\sigma \neq 0, \epsilon = \epsilon_r \epsilon_0, \mu = \mu_r \mu_0$)
4. Good conductors ($\sigma \approx \infty, \epsilon = \epsilon_0, \mu = \mu_r \mu_0$, or $\sigma \gg \omega \epsilon$)

where ω is the angular frequency of the wave. Case 3, for lossy dielectrics, is the most general case and will be considered first. Once this general case has been solved, we simply derive the other cases (1, 2, and 4) from it as special cases by changing the values of σ, μ , and ϵ . However, before we consider wave motion in those different media, it is appropriate that we study the characteristics of waves in general. This is important for proper understanding of EM waves. The reader who is conversant with the concept of waves may skip Section 10.2. Power considerations, reflection, and transmission between two different media will be discussed later in the chapter.

†10.2 WAVES IN GENERAL

A clear understanding of EM wave propagation depends on a grasp of what waves are in general. Wave motion occurs when a disturbance at point A, at time t_0 , is related to what happens at point

A **wave** is a function of both space and time.

B, at time $t > t_0$. A wave equation, as exemplified by eqs. (9.51) and (9.52), is a partial differential equation of the second order. In one dimension, a scalar wave equation takes the form of

$$\frac{\partial^2 E}{\partial t^2} - u^2 \frac{\partial^2 E}{\partial z^2} = 0 \quad (10.1)$$

where u is the *wave velocity*. Equation (10.1) is a special case of eq. (9.51) in which the medium is source free ($\rho_v = 0$, $\mathbf{J} = \mathbf{0}$). It can be solved by following a procedure similar to that in Example 6.5. Its solutions are of the form

$$E^+ = f(z - ut) \quad (10.2a)$$

$$E^- = g(z + ut) \quad (10.2b)$$

or

$$E = f(z - ut) + g(z + ut) \quad (10.2c)$$

where f and g denote any function of $z - ut$ and $z + ut$, respectively. Examples of such functions include $z \pm ut$, $\sin k(z \pm ut)$, $\cos k(z \pm ut)$, and $e^{jk(z \pm ut)}$, where k is a constant. It can easily be shown that these functions all satisfy eq. (10.1).

If we particularly assume harmonic (or sinusoidal) time dependence $e^{j\omega t}$, eq. (10.1) becomes (after replacing $\frac{\partial}{\partial t}$ by $j\omega$)

$$\frac{d^2 E_s}{dz^2} + \beta^2 E_s = 0 \quad (10.3)$$

where $\beta = \omega/u$ and E_s is the phasor form of E . The solution to eq. (10.3) is similar to Case 3 of Example 6.5 [see eq. (6.5.12)]. With the time factor inserted, the possible solutions to eq. (10.3) are

$$E^+ = Ae^{j(\omega t - \beta z)} \quad (10.4a)$$

$$E^- = Be^{j(\omega t + \beta z)} \quad (10.4b)$$

where E^+ means positive z -travel and E^- means negative travel. Combining E^+ and E^- leads to

$$E = Ae^{j(\omega t - \beta z)} + Be^{j(\omega t + \beta z)} \quad (10.4c)$$

where A and B are real constants.

For the moment, let us consider the solution in eq. (10.4a). Taking the imaginary part of this equation, we have

$$E = A \sin(\omega t - \beta z) \quad (10.5)$$

This is a sine wave chosen for simplicity; a cosine wave would have resulted had we taken the real part of eq. (10.4a). Note the following characteristics of the wave in eq. (10.5):

1. It is time harmonic because we assumed time dependence of the form $e^{j\omega t}$ to arrive at eq. (10.5).
2. The *amplitude* of the wave A has the same units as E .

3. The *phase* (in radians) of the wave depends on time t and space variable z ; it is the term $(\omega t - \beta z)$.
4. The *angular frequency* ω is given in radians per second; β , the *phase constant* or *wave number*, is given in radians per meter.

Because E varies with both time t and the space variable z , we may plot E as a function of t by keeping z constant and vice versa. The plots of $E(z, t = \text{constant})$ and $E(t, z = \text{constant})$ are shown in Figure 10.1(a) and (b), respectively. From Figure 10.1(a), we observe that the wave takes distance λ to repeat itself and hence λ is called the *wavelength* (in meters). From Figure 10.1(b), the wave takes time T to repeat itself; consequently T is known as the *period*, in seconds. Since it takes time T for the wave to travel distance λ at the speed u , we expect

$$\lambda = uT \quad (10.6a)$$

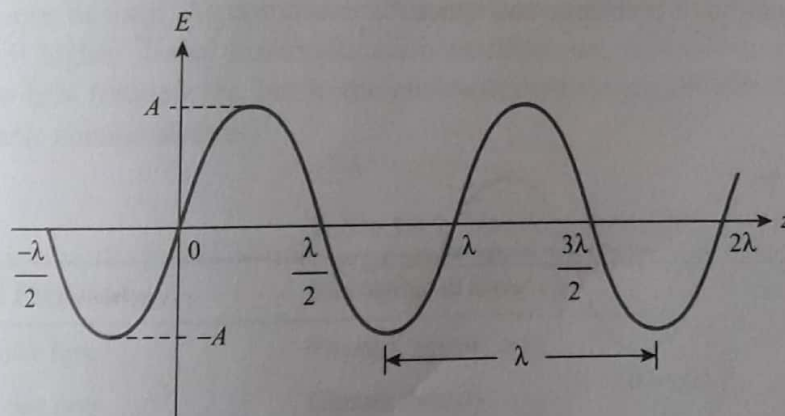
But $T = 1/f$, where f is the *frequency* (the number of cycles per second) of the wave in hertz (Hz). Hence,

$$u = f\lambda \quad (10.6b)$$

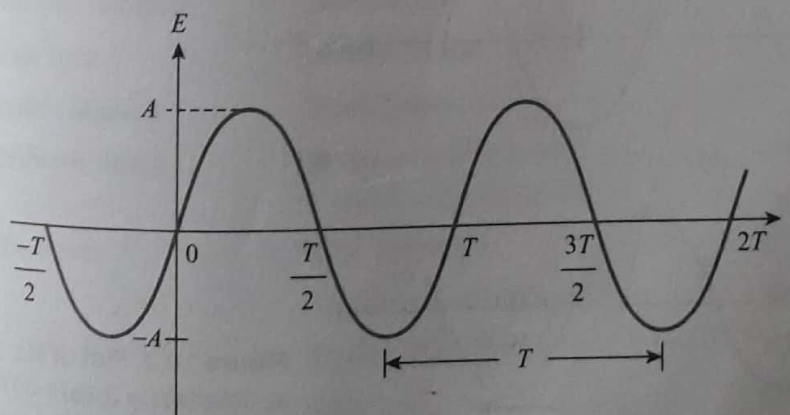
Because of this fixed relationship between wavelength and frequency, one can identify the position of a radio station within its band by either the frequency or the wavelength. Usually the frequency is preferred. Also, because

$$\omega = 2\pi f \quad (10.7a)$$

$$\beta = \frac{\omega}{u} \quad (10.7b)$$



(a)



(b)

Figure 10.1 Plot of $E(z, t) = A \sin(\omega t - \beta z)$:
(a) with constant t ,
(b) with constant z .

and

$$T = \frac{1}{f} = \frac{2\pi}{\omega} \quad (10.7)$$

we expect from eqs. (10.6) and (10.7) that

$$\beta = \frac{2\pi}{\lambda} \quad (10.8)$$

Equation (10.8) shows that for every wavelength of distance traveled, a wave undergoes a phase change of 2π radians.

We will now show that the wave represented by eq. (10.5) is traveling with a velocity u in the $+z$ -direction. To do this, we consider a fixed point P on the wave. We sketch eq. (10.5) at times $t = 0$, $T/4$, and $T/2$ as in Figure 10.2. From the figure, it is evident that as the wave advances with time, point P moves along $+z$ -direction. Point P is a point of constant phase, therefore

$$\omega t - \beta z = \text{constant}$$

or

$$\frac{dz}{dt} = \frac{\omega}{\beta} = u \quad (10.9)$$

which is the same as eq. (10.7b). Equation (10.9) shows that the wave travels with velocity u in the $+z$ -direction. Similarly, it can be shown that the wave $B \sin(\omega t + \beta z)$ in eq. (10.4b) is traveling with velocity u in the $-z$ -direction.

In summary, we note the following:

1. A wave is a function of both time and space.

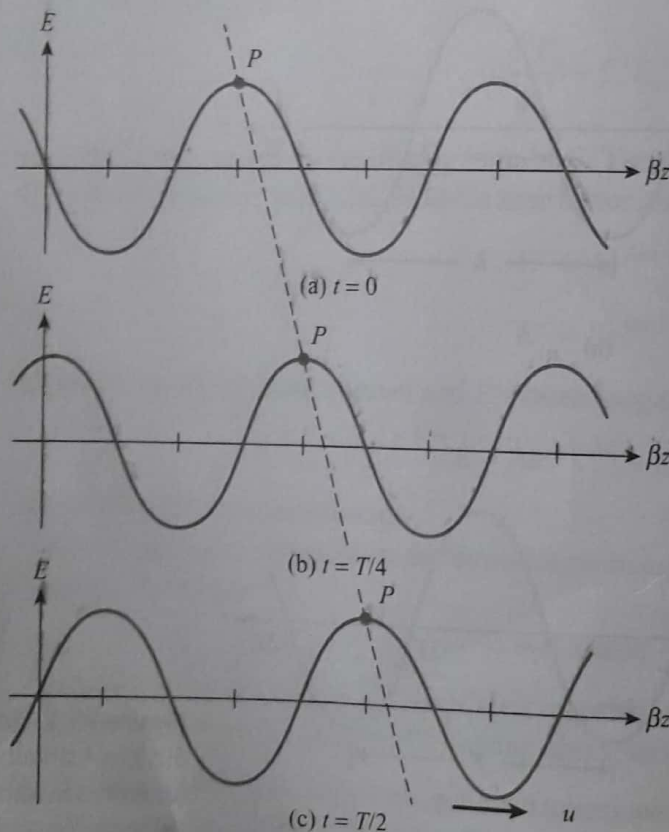


Figure 10.2 Plot of $E(z, t) = A \sin(\omega t - \beta z)$ at time (a) $t = 0$, (b) $t = T/4$, (c) $t = T/2$; P moves in the $+z$ -direction with velocity u .

2. Though time $t = 0$ is arbitrarily selected as a reference for the wave, a wave is without beginning or end.
3. A negative sign in $(\omega t \pm \beta z)$ is associated with a wave propagating in the $+z$ -direction (forward-traveling or positive-going wave), whereas a positive sign indicates that a wave is traveling in the $-z$ -direction (backward-traveling or negative-going wave).
4. Since $\sin(-\psi) = -\sin \psi = \sin(\psi \pm \pi)$, whereas $\cos(-\psi) = \cos \psi$,

$$\sin(\psi \pm \pi/2) = \pm \cos \psi \quad (10.10a)$$

$$\sin(\psi \pm \pi) = -\sin \psi \quad (10.10b)$$

$$\cos(\psi \pm \pi/2) = \mp \sin \psi \quad (10.10c)$$

$$\cos(\psi \pm \pi) = -\cos \psi \quad (10.10d)$$

where $\psi = \omega t \pm \beta z$. One of the relations in eqs. (10.10) can be used to represent any time-harmonic wave in the form of sine or cosine.

A large number of frequencies visualized in numerical order constitute a *spectrum*. Table 10.1 shows the frequencies at which various types of energy in the EM spectrum occur. Frequencies usable for radio communication occur near the lower end of the EM spectrum. As frequency increases, the manifestation of EM energy becomes dangerous to human beings. Microwave ovens, for example, can pose a hazard if not properly shielded. The practical difficulties of using EM energy for communication purposes also increase as frequency increases, until finally it can no longer be used. As communication methods improve, the limit to usable frequency has been pushed higher. Today communication satellites use frequencies near 14 GHz. This is still far below light frequencies, but in the enclosed environment of fiber optics, light itself can be used for radio communication.

Table 10.1 Electromagnetic Spectrum

EM Phenomena	Examples of Uses	Approximate Frequency Range
Cosmic rays	Physics, astronomy	10^{14} GHz and above
Gamma rays	Cancer therapy	10^{10} – 10^{13} GHz
X-rays	X-ray examination	10^8 – 10^9 GHz
Ultraviolet radiation	Sterilization	10^6 – 10^8 GHz
Visible light	Human vision	10^5 – 10^6 GHz
Infrared radiation	Photography	10^3 – 10^4 GHz
Microwave waves	Radar, microwave relays, satellite communication	3–300 GHz
Radio waves	UHF television	470–806 MHz
	VHF television, FM radio	54–216 MHz
	Short-wave radio	3–26 MHz
	AM radio	535–1605 kHz

10.3 WAVE PROPAGATION IN LOSSY DIELECTRICS

As mentioned in Section 10.1, wave propagation in lossy dielectrics is a general case from which wave propagation in media of other types can be derived as special cases. Therefore, this section is foundational to the next three sections.

A **lossy dielectric** is a medium in which an EM wave, as it propagates, loses power owing to imperfect dielectric.

In other words, a lossy dielectric is a partially conducting medium (imperfect dielectric or imperfect conductor) with $\sigma \neq 0$, as distinct from a lossless dielectric (perfect or good dielectric) in which $\sigma = 0$.

Consider a linear, isotropic, homogeneous, lossy dielectric medium that is charge free (macroscopic $\rho_v = 0$). Assuming and suppressing the time factor $e^{j\omega t}$, Maxwell's equations (see Table 9.2) become

$$\nabla \cdot \mathbf{E}_s = 0 \quad (10.11)$$

$$\nabla \cdot \mathbf{H}_s = 0 \quad (10.12)$$

$$\nabla \times \mathbf{E}_s = -j\omega\mu\mathbf{H}_s \quad (10.13)$$

$$\nabla \times \mathbf{H}_s = (\sigma + j\omega\epsilon)\mathbf{E}_s \quad (10.14)$$

Taking the curl of both sides of eq. (10.13) gives

$$\nabla \times \nabla \times \mathbf{E}_s = -j\omega\mu(\nabla \times \mathbf{H}_s) \quad (10.15)$$

Applying the vector identity

$$\nabla \times (\nabla \times \mathbf{A}) = \nabla(\nabla \cdot \mathbf{A}) - \nabla^2 \mathbf{A} \quad (10.16)$$

to the left-hand side of eq. (10.15) and invoking eqs. (10.11) and (10.14), we obtain

$$\nabla(\underbrace{\nabla \cdot \mathbf{E}_s}_0) - \nabla^2 \mathbf{E}_s = -j\omega\mu(\sigma + j\omega\epsilon)\mathbf{E}_s$$

or

$$\boxed{\nabla^2 \mathbf{E}_s - \gamma^2 \mathbf{E}_s = 0} \quad (10.17)$$

where

$$\gamma^2 = j\omega\mu(\sigma + j\omega\epsilon) \quad (10.18)$$

and γ , in reciprocal meters, is called the *propagation constant* of the medium. By a similar procedure, it can be shown that for the $\mathbf{H}$ field,

$$\nabla^2 \mathbf{H}_s - \gamma^2 \mathbf{H}_s = 0 \quad (10.19)$$

Equations (10.17) and (10.19) are known as homogeneous vector *Helmholtz's equations* or simply *vector wave equations*. In Cartesian coordinates, eq. (10.17), for example, is equivalent to three scalar wave equations, one for each component of $\mathbf{E}$ along $\mathbf{a}_x$, $\mathbf{a}_y$, and $\mathbf{a}_z$.

Since γ in eqs. (10.17) to (10.19) is a complex quantity, we may let

$$\gamma = \alpha + j\beta \quad (10.20)$$

We obtain α and β from eqs. (10.18) and (10.20) by noting that

$$-\text{Re } \gamma^2 = \beta^2 - \alpha^2 = \omega^2 \mu \epsilon \quad (10.21)$$

and

$$|\gamma^2| = \beta^2 + \alpha^2 = \omega \mu \sqrt{\sigma^2 + \omega^2 \epsilon^2} \quad (10.22)$$

From eqs. (10.21) and (10.22), we obtain

$$\alpha = \omega \sqrt{\frac{\mu \epsilon}{2} \left[\sqrt{1 + \left[\frac{\sigma}{\omega \epsilon} \right]^2} - 1 \right]} \quad (10.23)$$

$$\beta = \omega \sqrt{\frac{\mu \epsilon}{2} \left[\sqrt{1 + \left[\frac{\sigma}{\omega \epsilon} \right]^2} + 1 \right]} \quad (10.24)$$

For free space

$$u = \frac{\omega}{\beta} = \frac{\omega}{\omega \sqrt{\mu \epsilon}} = \frac{1}{\sqrt{\mu \epsilon}}$$

is independent of frequency. But for a lossy case, β is not a linear function of ω in eq. (10.24) and hence u is a function of ω . Furthermore, ϵ varies in a complex way with frequency for lossy dielectrics as explained in Appendix A. Therefore, different frequency components in a wave travel with different velocities causing what is known as *dispersion*.

$$\mathbf{E}_s = E_{xs}(z) \mathbf{a}_x \quad (10.25)$$

We then substitute into eq. (10.17), which yields

$$(\nabla^2 - \gamma^2) E_{xs}(z) = 0 \quad (10.26)$$

Without loss of generality, if we assume that a wave propagates in an unbounded medium along $\mathbf{a}_z$ and that $\mathbf{E}$ has only an x -component that does not vary with x and y , then

$$\frac{\partial^2 E_{xs}(z)}{\partial x^2} + \frac{\partial^2 E_{xs}(z)}{\partial y^2} + \frac{\partial^2 E_{xs}(z)}{\partial z^2} - \gamma^2 E_{xs}(z) = 0$$

or

$$\left[\frac{d^2}{dz^2} - \gamma^2 \right] E_{xs}(z) = 0 \quad (10.27)$$

This is a scalar wave equation, a linear homogeneous differential equation, with solution (see eq. 6.5.13a in Case 3 of Example 6.5),

$$E_{xs}(z) = E_0 e^{-\gamma z} + E_0' e^{\gamma z} \quad (10.28)$$

where E_o and E'_o are constants. The fact that the field must be finite at infinity requires that $E'_o = 0$. Alternatively, because $e^{\gamma z}$ denotes a wave traveling along $-z$, whereas we assume wave propagation along z , $E'_o = 0$. Whichever way we look at it, $E'_o = 0$. Inserting the time factor $e^{j\omega t}$ into eq. (10.28) and using eq. (10.20), we obtain

$$\mathbf{E}(z, t) = \text{Re}[E_{xs}(z)e^{j\omega t}\mathbf{a}_x] = \text{Re}(E_o e^{-\alpha z} e^{j(\omega t - \beta z)}\mathbf{a}_x)$$

or

$$\mathbf{E}(z, t) = E_o e^{-\alpha z} \cos(\omega t - \beta z)\mathbf{a}_x \quad (10.29)$$

A sketch of $|\mathbf{E}|$ at times $t = 0$ and $t = \Delta t$ is portrayed in Figure 10.5, where it is evident that $\mathbf{E}$ has only an x -component and it is traveling in the $+z$ -direction. Having obtained $\mathbf{E}(z, t)$, we obtain $\mathbf{H}(z, t)$ either by taking similar steps to solve eq. (10.19) or by using eq. (10.29) in conjunction with Maxwell's equations, as we did in Example 9.8. We will eventually arrive at

$$\mathbf{H}(z, t) = \text{Re}(H_o e^{-\alpha z} e^{j(\omega t - \beta z)}\mathbf{a}_y) \quad (10.30)$$

where

$$H_o = \frac{E_o}{\eta} \quad (10.31)$$

and η is a complex quantity known as the *intrinsic impedance*, in ohms, of the medium. It can be shown by following the steps taken in Example 9.8 that

$$\eta = \sqrt{\frac{j\omega\mu}{\sigma + j\omega\epsilon}} = |\eta| \angle \theta_\eta = |\eta| e^{j\theta_\eta} \quad (10.32)$$

with

$$|\eta| = \frac{\sqrt{\mu/\epsilon}}{\left[1 + \left(\frac{\sigma}{\omega\epsilon}\right)^2\right]^{1/4}}, \quad \tan 2\theta_\eta = \frac{\sigma}{\omega\epsilon} \quad (10.33)$$

where $0 \leq \theta_\eta \leq 45^\circ$. Substituting eqs. (10.31) and (10.32) into eq. (10.30) gives

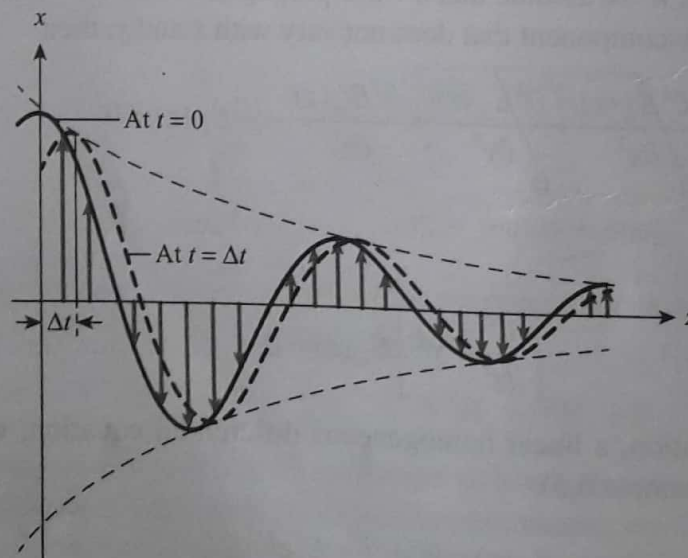


Figure 10.5 An E -field with an x -component traveling in the $+z$ -direction at times $t = 0$ and $t = \Delta t$; arrows indicate instantaneous values of E .

$$\mathbf{H} = \text{Re} \left[\frac{E_0}{|\eta| e^{j\theta_\eta}} e^{-\alpha z} e^{j(\omega t - \beta z)} \mathbf{a}_y \right]$$

or

$$\mathbf{H} = \frac{E_0}{|\eta|} e^{-\alpha z} \cos(\omega t - \beta z - \theta_\eta) \mathbf{a}_y \quad (10.34)$$

Notice from eqs. (10.29) and (10.34) that as the wave propagates along $\mathbf{a}_z$, it decreases or attenuates in amplitude by a factor $e^{-\alpha z}$, and hence α is known as the *attenuation constant*, or attenuation coefficient, of the medium. It is a measure of the spatial rate of decay of the wave in the medium, measured in nepers per meter (Np/m) and can be expressed in decibels per meter (dB/m). An attenuation of 1 neper denotes a reduction to e^{-1} of the original value, whereas an increase of 1 neper indicates an increase by a factor of e . Hence, for voltages

$$1 \text{ Np} = 20 \log_{10} e = 8.686 \text{ dB} \quad (10.35)$$

From eq. (10.23), we notice that if $\sigma = 0$, as is the case for a lossless medium and free space, $\alpha = 0$ and the wave is not attenuated as it propagates. The quantity β is a measure of the phase shift per unit length in radians per meter and is called the *phase constant* or *wave number*. In terms of β , the wave velocity u and wavelength λ are, respectively, given by [see eqs. (10.7b) and (10.8)]

$$u = \frac{\omega}{\beta}, \quad \lambda = \frac{2\pi}{\beta} \quad (10.36)$$

We also notice from eqs. (10.29) and (10.34) that $\mathbf{E}$ and $\mathbf{H}$ are out of phase by θ_η at any instant of time due to the complex intrinsic impedance of the medium. Thus at any time, $\mathbf{E}$ leads $\mathbf{H}$ (or $\mathbf{H}$ lags $\mathbf{E}$) by θ_η . Finally, we notice that the ratio of the magnitude of the conduction current density $\mathbf{J}_c$ to that of the displacement current density $\mathbf{J}_d$ in a lossy medium is

$$\frac{|\mathbf{J}_{cs}|}{|\mathbf{J}_{ds}|} = \frac{|\sigma \mathbf{E}_s|}{|j\omega \epsilon \mathbf{E}_s|} = \frac{\sigma}{\omega \epsilon} = \tan \theta$$

or

$$\tan \theta = \frac{\sigma}{\omega \epsilon} \quad (10.37)$$

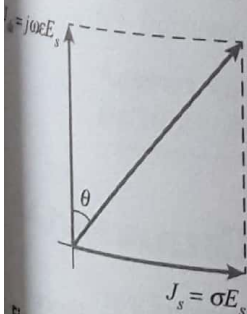


Figure 10.6 Loss angle of a lossy medium.

where $\tan \theta$ is known as the *loss tangent* and θ is the *loss angle* of the medium as illustrated in Figure 10.6. Although a line of demarcation between good conductors and lossy dielectrics is not easy to make, $\tan \theta$ or θ may be used to determine how lossy a medium is. A medium is said to be a good (lossless or perfect) dielectric if $\tan \theta$ is very small ($\sigma \ll \omega \epsilon$) or a good conductor if $\tan \theta$ is very large ($\sigma \gg \omega \epsilon$). From the viewpoint of wave propagation, the characteristic behavior of a medium depends not only on its constitutive parameters σ , ϵ , and μ but also on the frequency of operation. A medium that is regarded as a good conductor at low frequencies may be a good dielectric at high frequencies as explained in Appendix A. Note from eqs. (10.33) and (10.37) that

$$\theta = 2\theta_\eta$$

From eq. (10.14)

$$\begin{aligned}\nabla \times \mathbf{H}_s &= (\sigma + j\omega\epsilon)\mathbf{E}_s = j\omega\epsilon \left[1 - \frac{j\sigma}{\omega\epsilon} \right] \mathbf{E}_s \\ &= j\omega\epsilon_c \mathbf{E}_s\end{aligned}$$

Where

$$\epsilon_c = \epsilon \left[1 - j \frac{\sigma}{\omega\epsilon} \right] = \epsilon [1 - j \tan \theta]$$

or

$$\epsilon_c = \epsilon' - j\epsilon''$$

with $\epsilon' = \epsilon$, $\epsilon'' = \sigma/\omega$, $\epsilon = \epsilon_0 \epsilon_r$; ϵ_c is called the *complex permittivity* of the medium. We observe that the ratio of ϵ'' to ϵ' is the loss tangent of the medium, that is,

$$\tan \theta = \frac{\epsilon''}{\epsilon'} = \frac{\sigma}{\omega\epsilon}$$

In subsequent sections, we will consider wave propagation in media of other types that may be regarded as special cases of what we have considered here. Thus we will simply deduce the governing formulas from those obtained for the general case treated in this section. The student is advised not just to memorize the formulas but to observe how they are easily obtained from the formulas for the general case.

10.4 PLANE WAVES IN LOSSLESS DIELECTRICS

In a lossless dielectric, $\sigma \ll \omega\epsilon$. It is a special case of that in Section 10.3 except that

$$\sigma \approx 0, \quad \epsilon = \epsilon_0 \epsilon_r, \quad \mu = \mu_0 \mu_r$$

(10.42)

Substituting these into eqs. (10.23) and (10.24) gives

$$\alpha = 0, \quad \beta = \omega\sqrt{\mu\epsilon} \quad (10.43a)$$

$$u = \frac{\omega}{\beta} = \frac{1}{\sqrt{\mu\epsilon}}, \quad \lambda = \frac{2\pi}{\beta} \quad (10.43b)$$

Also

$$\eta = \sqrt{\frac{\mu}{\epsilon}} \angle 0^\circ \quad (10.44)$$

and thus **E** and **H** are in time phase with each other.

10.5 PLANE WAVES IN FREE SPACE

Plane waves in free space comprise a special case of what we considered in Section 10.3. In this case,

$$\sigma = 0, \quad \epsilon = \epsilon_0, \quad \mu = \mu_0 \quad (10.45)$$

This may also be regarded as a special case of Section 10.4. Thus we simply replace ϵ by ϵ_0 and μ by μ_0 in eq. (10.43) or we substitute eq. (10.45) directly into eqs. (10.23) and (10.24). Either way, we obtain

$$\alpha = 0, \quad \beta = \omega\sqrt{\mu_0\epsilon_0} = \frac{\omega}{c} \quad (10.46a)$$

$$u = \frac{1}{\sqrt{\mu_0\epsilon_0}} = c, \quad \lambda = \frac{2\pi}{\beta} \quad (10.46b)$$

where $c \approx 3 \times 10^8$ m/s, the speed of light in a vacuum. The fact that EM waves travel in free space at the speed of light is significant. It provides some evidence that light is the manifestation of an EM wave. In other words, light is characteristically electromagnetic.

By substituting the constitutive parameters in eq. (10.45) into eq. (10.33), $\theta_\eta = 0$ and $\eta = \eta_0$, where η_0 is called the *intrinsic impedance of free space* and is given by

$$\eta_0 = \sqrt{\frac{\mu_0}{\epsilon_0}} = 120\pi \approx 377 \Omega \quad (10.47)$$

$$\mathbf{E} = E_0 \cos(\omega t - \beta z) \mathbf{a}_x \quad (10.48a)$$

then

$$\mathbf{H} = H_0 \cos(\omega t - \beta z) \mathbf{a}_y = \frac{E_0}{\eta_0} \cos(\omega t - \beta z) \mathbf{a}_y \quad (10.48b)$$

The plots of **E** and **H** are shown in Figure 10.7(a). In general, if $\mathbf{a}_E$, $\mathbf{a}_H$, and $\mathbf{a}_k$ are unit vectors along the **E** field, the **H** field, and the direction of wave propagation; it can be shown that (see Problem 10.67).

$$\mathbf{a}_k \times \mathbf{a}_E = \mathbf{a}_H$$

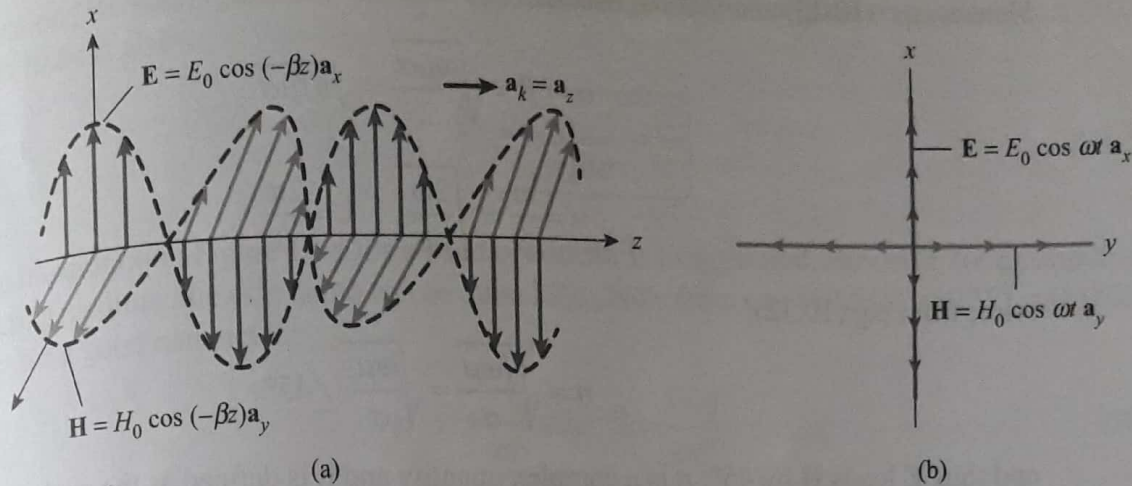


Figure 10.7 Plots of $\mathbf{E}$ and $\mathbf{H}$ (a) as functions of z at $t=0$; and (b) at $z=0$. The arrows indicate instantaneous values.

or

$$\mathbf{a}_k \times \mathbf{a}_H = -\mathbf{a}_E$$

or

$$\boxed{\mathbf{a}_E \times \mathbf{a}_H = \mathbf{a}_k} \quad (10.49)$$

Both $\mathbf{E}$ and $\mathbf{H}$ fields (or EM waves) are everywhere normal to the direction of wave propagation, $\mathbf{a}_k$. That means that the fields lie in a plane that is transverse or orthogonal to the direction of wave propagation. They form an EM wave that has no electric or magnetic field components along the direction of propagation; such a wave is called a *transverse electromagnetic* (TEM) wave. A combination of $\mathbf{E}$ and $\mathbf{H}$ is called a *uniform plane wave* because $\mathbf{E}$ (or $\mathbf{H}$) has the same magnitude throughout any transverse plane, defined by $z = \text{constant}$. The direction in which the electric field points is the *polarization* of a TEM wave.¹ The wave in eq. (10.29), for example, is polarized in the x -direction. This should be observed in Figure 10.7(b), which illustrates uniform plane waves. A uniform plane wave cannot exist physically because it stretches to infinity and would represent an infinite energy. Such waves are characteristically simple and fundamentally important. They serve as approximations to practical waves such as those from a radio antenna at distances sufficiently far from radiating sources. Although our discussion after eq. (10.48) deals with free space, it also applies for any other isotropic medium.

10.6 PLANE WAVES IN GOOD CONDUCTORS

Plane waves in good conductors comprise another special case of that considered in Section 10.3.

A perfect, or good conductor, is one in which $\sigma \gg \omega\epsilon$ so that $\frac{\sigma}{\omega\epsilon} \gg 1$; that is,

$$\boxed{\sigma = \infty, \mu = \mu_0 \mu_r} \quad (10.50)$$

¹Polarization will be covered in Section 10.7.

Hence, eqs. (10.23) and (10.24) become

$$\alpha = \beta = \sqrt{\frac{\omega\mu\sigma}{2}} = \sqrt{\pi f\mu\sigma} \quad (10.51a)$$

$$u = \frac{\omega}{\beta} = \sqrt{\frac{2\omega}{\mu\sigma}}, \quad \lambda = \frac{2\pi}{\beta} \quad (10.51b)$$

Also, from eq. (10.32),

$$\eta = \sqrt{\frac{j\omega\mu}{\sigma}} = \sqrt{\frac{\omega\mu}{\sigma}} \angle 45^\circ \quad (10.52)$$

and thus $\mathbf{E}$ leads $\mathbf{H}$ by 45° . η is a complex quantity and it is defined as the surface impedance for a good conductor.

$$\mathbf{E} = E_0 e^{-\alpha z} \cos(\omega t - \beta z) \mathbf{a}_x \quad (10.53a)$$

then

$$\mathbf{H} = \frac{E_0}{\sqrt{\frac{\omega\mu}{\sigma}}} e^{-\alpha z} \cos(\omega t - \beta z - 45^\circ) \mathbf{a}_y \quad (10.53b)$$

Therefore, as the $\mathbf{E}$ (or $\mathbf{H}$) wave travels in a conducting medium, its amplitude is attenuated by the factor $e^{-\alpha z}$. The distance δ , shown in Figure 10.8, through which the wave amplitude decreases to a factor e^{-1} (about 37% of the original value) is called *skin depth* or *penetration depth* of the medium; that is,

$$E_0 e^{-\alpha\delta} = E_0 e^{-1}$$

or

$$\delta = \frac{1}{\alpha} \quad (10.54a)$$

The **skin depth** is a measure of the depth to which an **EM** wave can penetrate the medium.

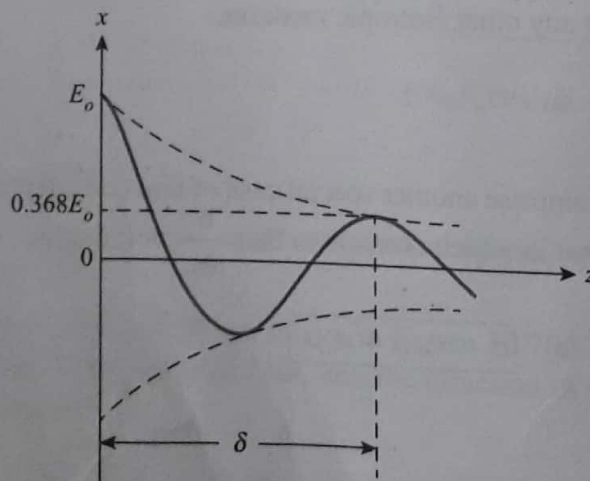


Figure 10.8 Illustration of skin depth.

Equation (10.54a) is generally valid for any material medium. For good conductors, eqs. (10.51a) and (10.54a) give

$$\delta = \frac{1}{\sqrt{\pi f \mu \sigma}} = \frac{1}{\alpha} \quad (10.54b)$$

The illustration in Figure 10.8 for a good conductor is exaggerated. However, for a partially conducting medium, the skin depth can be quite large. Note from eqs. (10.51a), (10.52), and (10.54b) that for a good conductor,

$$\eta = \frac{1}{\sigma \delta} \sqrt{2} e^{j\pi/4} = \frac{1+j}{\sigma \delta} \quad (10.55)$$

Noting that for good conductors we have $\alpha = \beta = \frac{1}{\delta}$, eq. (10.53a) can be written as

$$\mathbf{E} = E_0 e^{-z/\delta} \cos\left(\omega t - \frac{z}{\delta}\right) \mathbf{a}_x$$

showing that δ measures the exponential damping of the wave as it travels through the conductor. The skin depth in copper at various frequencies is shown in Table 10.2. From the table, we notice that the skin depth decreases with increasing frequency. Thus, $\mathbf{E}$ and $\mathbf{H}$ can hardly propagate through good conductors.

Table 10.2 Skin Depth in Copper*

Frequency (Hz)	10	60	100	500	10^4	10^8	10^{10}
Skin depth (mm)	20.8	8.6	6.6	2.99	0.66	6.6×10^{-3}	6.6×10^{-4}

*For copper, $\sigma = 5.8 \times 10^7$ S/m, $\mu = \mu_0$, $\delta = 66.1 \sqrt{f}$ (in mm).

The phenomenon whereby field intensity in a conductor rapidly decreases is known as the *skin effect*. It is a tendency of charges to migrate from the bulk of the conducting material to the surface, resulting in higher resistance. The fields and associated currents are confined to a very thin layer (the skin) of the conductor surface. For a wire of radius a , for example, it is a good approximation at high frequencies to assume that all of the current flows in the circular ring of thickness δ as shown in Figure 10.9. The skin effect appears in different guises in such problems as attenuation in waveguides, effective or ac resistance of transmission lines, and electromagnetic shielding. It is used to advantage in many applications. For example, because the skin depth in

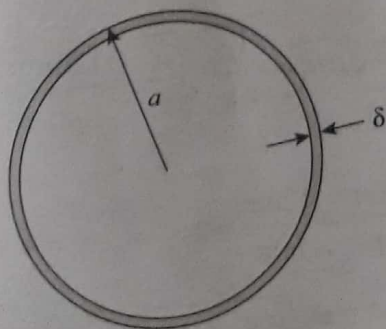


Figure 10.9 Skin depth at high frequencies, $\delta \ll a$.

silver is very small, the difference in performance between a pure silver component and a silver-plated brass component is negligible, so silver plating is often used to reduce the material cost of waveguide components. For the same reason, hollow tubular conductors are used instead of solid conductors in outdoor television antennas. Effective electromagnetic shielding of electrical devices can be provided by conductive enclosures a few skin depths in thickness.

The skin depth is useful in calculating the *ac resistance* due to skin effect. The resistance in eq. (5.16) is called the *dc resistance*, that is,

$$R_{dc} = \frac{\ell}{\sigma S}$$

We define the *surface or skin resistance* R_s (in Ω) as the real part of η for a good conductor. Thus from eq. (10.55)

$$R_s = \frac{1}{\sigma \delta} = \sqrt{\frac{\pi f \mu}{\sigma}} \quad (10.56)$$

This is the resistance of a unit width and unit length of the conductor. It is equivalent to the dc resistance for a unit length of the conductor having cross-sectional area $1 \times \delta$. Thus for a given width w and length ℓ , the ac resistance is calculated by using the familiar dc resistance relation of eq. (5.16) and assuming a uniform current flow in the conductor of thickness δ , that is,

$$R_{ac} = \frac{\ell}{\sigma \delta w} = \frac{R_s \ell}{w} \quad (10.57)$$

where $S \approx \delta w$. For a conductor wire of radius a (see Figure 10.9), $w = 2\pi a$, so

$$\frac{R_{ac}}{R_{dc}} = \frac{\frac{\ell}{\sigma 2\pi a \delta}}{\frac{\ell}{\sigma \pi a^2}} = \frac{a}{2\delta} = \frac{a}{2} \sqrt{\pi f \mu \sigma}$$

Since $\delta \ll a$ at high frequencies, this shows that R_{ac} is far greater than R_{dc} . In general, the ratio of the ac to the dc resistance starts at 1.0 for dc and very low frequencies and increases as the frequency increases. Also, although the bulk of the current is nonuniformly distributed over a thickness of 5δ of the conductor, the power loss is the same as though it were uniformly distributed over a thickness of δ and zero elsewhere. This is one more reason that δ is referred to as the skin depth.

10.8 POWER AND THE POYNTING VECTOR

As mentioned before, energy can be transported from one point (where a transmitter is located) to another point (with a receiver) by means of EM waves. The rate of such energy transportation can be obtained from Maxwell's equations:

$$\nabla \times \mathbf{E} = -\mu \frac{\partial \mathbf{H}}{\partial t} \quad (10.68a)$$

$$\nabla \times \mathbf{H} = \sigma \mathbf{E} + \epsilon \frac{\partial \mathbf{E}}{\partial t} \quad (10.68b)$$

Dotting both sides of eq. (10.68b) with $\mathbf{E}$ gives

$$\mathbf{E} \cdot (\nabla \times \mathbf{H}) = \sigma E^2 + \mathbf{E} \cdot \epsilon \frac{\partial \mathbf{E}}{\partial t} \quad (10.69)$$

But for any vector fields $\mathbf{A}$ and $\mathbf{B}$ (see Appendix B.10)

$$\nabla \cdot (\mathbf{A} \times \mathbf{B}) = \mathbf{B} \cdot (\nabla \times \mathbf{A}) - \mathbf{A} \cdot (\nabla \times \mathbf{B})$$

Applying this vector identity to eq. (10.69) (letting $\mathbf{A} = \mathbf{H}$ and $\mathbf{B} = \mathbf{E}$) gives

$$\begin{aligned} \mathbf{H} \cdot (\nabla \times \mathbf{E}) + \nabla \cdot (\mathbf{H} \times \mathbf{E}) &= \sigma E^2 + \mathbf{E} \cdot \epsilon \frac{\partial \mathbf{E}}{\partial t} \\ &= \sigma E^2 + \frac{1}{2} \epsilon \frac{\partial}{\partial t} E^2 \end{aligned} \quad (10.70)$$

since $\frac{\partial E^2}{\partial t} = \frac{\partial}{\partial t}(\mathbf{E} \cdot \mathbf{E}) = \mathbf{E} \cdot \frac{\partial \mathbf{E}}{\partial t} + \mathbf{E} \cdot \frac{\partial \mathbf{E}}{\partial t} = 2\mathbf{E} \cdot \frac{\partial \mathbf{E}}{\partial t}$. Dotting both sides of eq. (10.68a) with $\mathbf{H}$, we write

$$\mathbf{H} \cdot (\nabla \times \mathbf{E}) = \mathbf{H} \cdot \left(-\mu \frac{\partial \mathbf{H}}{\partial t} \right) = -\frac{\mu}{2} \frac{\partial}{\partial t} (\mathbf{H} \cdot \mathbf{H}) \quad (10.71)$$

and thus eq. (10.70) becomes

$$-\frac{\mu}{2} \frac{\partial H^2}{\partial t} - \nabla \cdot (\mathbf{E} \times \mathbf{H}) = \sigma E^2 + \frac{1}{2} \epsilon \frac{\partial E^2}{\partial t}$$

Rearranging terms and taking the volume integral of both sides,

$$\int_v \nabla \cdot (\mathbf{E} \times \mathbf{H}) dv = -\frac{\partial}{\partial t} \int_v \left[\frac{1}{2} \epsilon E^2 + \frac{1}{2} \mu H^2 \right] dv - \int_v \sigma E^2 dv \quad (10.72)$$

Applying the divergence theorem to the left-hand side gives

$$\oint_s (\mathbf{E} \times \mathbf{H}) \cdot d\mathbf{S} = -\frac{\partial}{\partial t} \int_v \left[\frac{1}{2} \epsilon E^2 + \frac{1}{2} \mu H^2 \right] dv - \int_v \sigma E^2 dv \quad (10.73)$$

$$\begin{array}{ccccc} \downarrow & & \downarrow & & \downarrow \\ \text{total power} & & \text{rate of decrease in energy} & & \text{ohmic power} \\ \text{leaving the} & = & \text{stored in electric} & - & \text{dissipated} \\ \text{volume} & & \text{and magnetic fields} & & \end{array} \quad (10.74)$$

Equation (10.73) is referred to as *Poynting's theorem*.² The various terms in the equation are identified using energy-conservation arguments for EM fields. The first term on the right-hand side of eq. (10.73) is interpreted as the rate of decrease in energy stored in the electric and magnetic fields. The second term is the power dissipated because the medium is conducting ($\sigma \neq 0$). The quantity $\mathbf{E} \times \mathbf{H}$ on the left-hand side of eq. (10.73) is known as the *Poynting vector* $\mathcal{P}$, measured in watts per square meter (W/m^2); that is,

$$\boxed{\mathcal{P} = \mathbf{E} \times \mathbf{H}} \quad (10.75)$$

It represents the instantaneous power density vector associated with the EM field at a given point. The integration of the Poynting vector over any closed surface gives the net power flowing out of that surface.

Poynting's theorem states that the net power flowing out of a given volume v is equal to the time rate of decrease in the energy stored within v minus the ohmic losses.

The theorem is illustrated in Figure 10.11.

It should be noted that $\mathcal{P}$ is normal to both $\mathbf{E}$ and $\mathbf{H}$ and is therefore along the direction of wave propagation $\mathbf{a}_k$ for uniform plane waves. Thus

$$\mathbf{a}_k = \mathbf{a}_E \times \mathbf{a}_H \quad (10.49)$$

The fact that $\mathcal{P}$ points along $\mathbf{a}_k$ causes $\mathcal{P}$ to be regarded as a "pointing" vector.

²After J. H. Poynting, "On the transfer of energy in the electromagnetic field," *Philosophical Transactions*, vol. 174, 1883, p. 343.

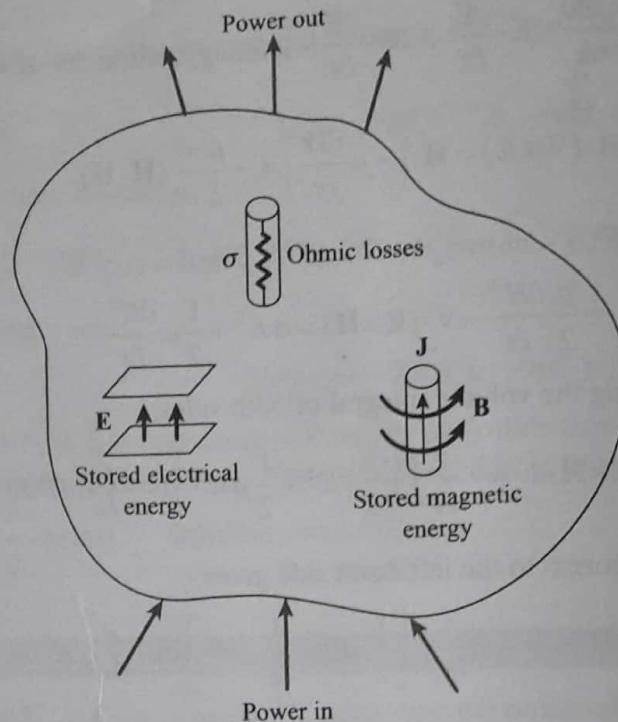


Figure 10.11 Illustration of power balance for EM fields.

Again, if we assume that

$$\mathbf{E}(z, t) = E_0 e^{-az} \cos(\omega t - \beta z) \mathbf{a}_x$$

then

$$\mathbf{H}(z, t) = \frac{E_0}{|\eta|} e^{-az} \cos(\omega t - \beta z - \theta_\eta) \mathbf{a}_y$$

and

$$\begin{aligned} \mathcal{P}(z, t) &= \frac{E_0^2}{|\eta|} e^{-2az} \cos(\omega t - \beta z) \cos(\omega t - \beta z - \theta_\eta) \mathbf{a}_z \\ &= \frac{E_0^2}{2|\eta|} e^{-2az} [\cos \theta_\eta + \cos(2\omega t - 2\beta z - \theta_\eta)] \mathbf{a}_z \end{aligned} \quad (10.76)$$

since $\cos A \cos B = \frac{1}{2} [\cos(A - B) + \cos(A + B)]$. To determine the time-average Poynting vector $\mathcal{P}_{\text{ave}}(z)$ (in W/m^2), which is of more practical value than the instantaneous Poynting vector $\mathcal{P}(z, t)$, we integrate eq. (10.76) over the period $T = 2\pi/\omega$; that is,

$$\mathcal{P}_{\text{ave}}(z) = \frac{1}{T} \int_0^T \mathcal{P}(z, t) dt \quad (10.77)$$

It can be shown (see Problem 10.40) that this is equivalent to

$$\boxed{\mathcal{P}_{\text{ave}}(z) = \frac{1}{2} \text{Re}(\mathbf{E}_s \times \mathbf{H}_s^*)} \quad (10.78)$$

By substituting eq. (10.76) into eq. (10.77), we obtain

$$\mathcal{P}_{\text{ave}}(z) = \frac{E_o^2}{2|\eta|} e^{-2\alpha z} \cos\theta_\eta \mathbf{a}_z \quad (10.79)$$

The total time-average power crossing a given surface S is given by

$$P_{\text{ave}} = \int_S \mathcal{P}_{\text{ave}} \cdot d\mathbf{S} \quad (10.80)$$

We should note the difference between $\mathcal{P}$, $\mathcal{P}_{\text{ave}}$, and P_{ave} : whereas $\mathcal{P}(x, y, z, t)$ is the Poynting vector in watts per square meter and is time varying, $\mathcal{P}_{\text{ave}}(x, y, z)$ also in watts per square meter, is the time average of the Poynting vector $\mathcal{P}$; it is a vector but is time invariant. Finally, P_{ave} is a total time-average power through a surface in watts; it is a scalar.