5.4.3 Evaluation of Integrals Over a Given Region in Polar Coordinates

Example 1

Evaluate $\iint r\sqrt{a^2 - r^2}\,dr\,d\theta$, *over the upper half of the circle* $r = a \cos\theta$.

Solution

1. The region of integration is the upper half of the circle $r = a \cos \theta$.
2. Draw an elementary radius vector OA which starts from the origin and terminates on the circle $r = a \cos\theta$.

   Limits of $r : r = 0$ to $r = a \cos\theta$

   Limits of $\theta : \theta = 0$ to $\theta = \dfrac{\pi}{2}$

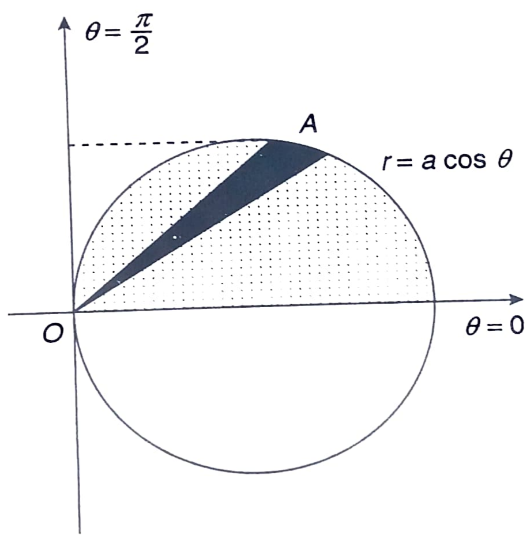

Fig. 5.47

$$I = \iint r\sqrt{a^2 - r^2}\,dr\,d\theta$$

$$= \int_0^{\frac{\pi}{2}} \int_0^{a\cos\theta} \left(-\frac{1}{2}\right)(a^2 - r^2)^{\frac{1}{2}}(-2r)\,dr\,d\theta$$

$$= -\frac{1}{2}\int_0^{\frac{\pi}{2}} \left| \frac{2(a^2 - r^2)^{\frac{3}{2}}}{3} \right|_0^{a\cos\theta} d\theta \qquad \left[\because \int [f(r)]^n f'(r)\,dr = \frac{[f(r)]^{n+1}}{n+1}\right]$$

$$= -\frac{1}{3}\int_0^{\frac{\pi}{2}} (a^3 \sin^3\theta - a^3)\,d\theta$$

$$= -\frac{a^3}{3}\int_0^{\frac{\pi}{2}} \left(\frac{3\sin\theta - \sin 3\theta}{4} - 1\right) d\theta$$

$$= -\frac{a^3}{3}\left|\frac{1}{4}\left(-3\cos\theta + \frac{\cos 3\theta}{3}\right) - \theta\right|_0^{\frac{\pi}{2}}$$

$$= -\frac{a^3}{3}\left(-\frac{3}{4}\cos\frac{\pi}{2} + \frac{1}{12}\cos\frac{3\pi}{2} - \frac{\pi}{2} + \frac{3}{4}\cos 0 - \frac{1}{12}\cos 0\right)$$

$$= -\frac{a^3}{3}\left(-\frac{\pi}{2} + \frac{3}{4} - \frac{1}{12}\right)$$

$$= -\frac{a^3}{3}\left(\frac{2}{3} - \frac{\pi}{2}\right)$$

Example 2

Evaluate $\iint r^4 \cos^3\theta \, dr\, d\theta$, *over the interior of the circle* $r = 2a\cos\theta$.

Solution

1. The region of integration is the interior of the circle $r = 2a\cos\theta$.
2. Draw an elementary radius vector OA which starts from the origin and terminates on the circle $r = 2a\cos\theta$.

Limits of $r : r = 0$ to $r = 2a\cos\theta$

Limits of $\theta : \theta = -\frac{\pi}{2}$ to $\theta = \frac{\pi}{2}$

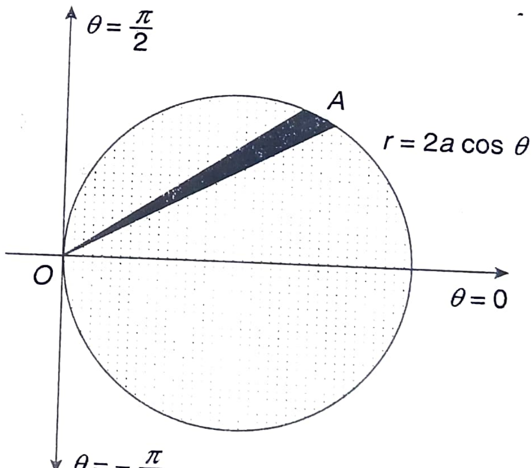

Fig. 5.48

$$I = \iint r^4 \cos^3\theta \, dr\, d\theta$$

$$= \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \int_0^{2a\cos\theta} r^4 \cos^3\theta \, dr\, d\theta$$

$$= \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \left|\frac{r^5}{5}\right|_0^{2a\cos\theta} \cos^3\theta \, d\theta$$

$$= \frac{1}{5}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} (2a\cos\theta)^5 \cos^3\theta \, d\theta$$

$$= \frac{32a^5}{5} \cdot 2\int_0^{\frac{\pi}{2}} \cos^8\theta \, d\theta \qquad [\because \cos^8(-\theta) = \cos^8\theta]$$

$$= \frac{64a^5}{5}\left(\frac{7}{8}\cdot\frac{5}{6}\cdot\frac{3}{4}\cdot\frac{1}{2}\cdot\frac{\pi}{2}\right) \qquad \text{[Using reduction formula]}$$

$$= \frac{7\pi}{4}a^5$$

Example 3

Evaluate $\iint r^2 \sin\theta \, dr\, d\theta$, *over the cardioid* $r = a(1 + \cos\theta)$ *above the initial line.*

Solution

1. The region of integration is the part of the cardioid $r = a(1 + \cos\theta)$ above the initial line ($\theta = 0$).
2. Draw an elementary radius vector OA which starts from the origin and terminates on the cardioid $r = a(1 + \cos\theta)$.
3. Limits of r: $r = 0$ to $r = a(1 + \cos\theta)$
   Limits of θ: $\theta = 0$ to $\theta = \pi$

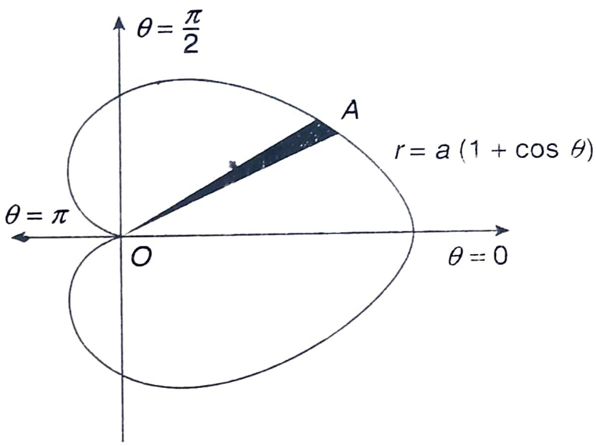

Fig. 5.49

$$
\begin{aligned}
I &= \iint r^2 \sin\theta \, dr \, d\theta \\
&= \int_0^{\pi} \int_0^{a(1+\cos\theta)} r^2 \sin\theta \, dr \, d\theta \\
&= \int_0^{\pi} \left| \frac{r^3}{3} \right|_0^{a(1+\cos\theta)} \sin\theta \, d\theta \\
&= \frac{1}{3} \int_0^{\pi} a^3 (1+\cos\theta)^3 \sin\theta \cdot d\theta \\
&= -\frac{a^3}{3} \int_0^{\pi} (1+\cos\theta)^3 (-\sin\theta) d\theta \\
&= -\frac{a^3}{3} \left| \frac{(1+\cos\theta)^4}{4} \right|_0^{\pi} \qquad \left[\because \int [f(\theta)]^n f'(\theta) d\theta = \frac{[f(\theta)]^{n+1}}{n+1} \right] \\
&= -\frac{a^3}{12} \left[(1+\cos\pi)^4 - (1+\cos 0)^4 \right] \\
&= -\frac{a^3}{12} (0 - 16) \\
&= \frac{4}{3} a^2
\end{aligned}
$$

Example 4

Evaluate $\iint \frac{r \, dr \, d\theta}{\sqrt{r^2 + a^2}}$, *over one loop of the lemniscate* $r^2 = a^2 \cos 2\theta$.

Solution

1. The region of integration is one loop of the lemniscate $r^2 = a^2 \cos 2\theta$ bounded between the lines $\theta = -\frac{\pi}{4}$ and $\theta = \frac{\pi}{4}$.
2. Draw an elementary radius vector OA which starts from the origin and terminates on the lemniscate $r^2 = a^2 \cos 2\theta$.
3. Limits of r: $r = 0$ to $r = a\sqrt{\cos 2\theta}$

   Limits of θ: $\theta = -\frac{\pi}{4}$ to $\theta = \frac{\pi}{4}$

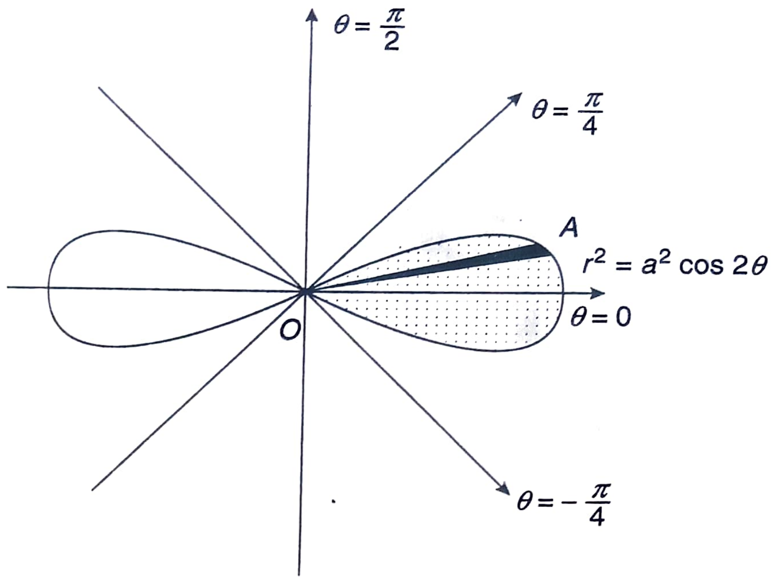

Fig. 5.50

$$I=\iint \frac{r\,dr\,d\theta}{\sqrt{r^2+a^2}}$$

$$=\int_{-\frac{\pi}{4}}^{\frac{\pi}{4}}\int_0^{a\sqrt{\cos 2\theta}} \frac{r\,dr\,d\theta}{\sqrt{r^2+a^2}}$$

$$=\int_{-\frac{\pi}{4}}^{\frac{\pi}{4}}\int_0^{a\sqrt{\cos 2\theta}} \frac{1}{2}(r^2+a^2)^{-\frac{1}{2}}(2r)dr\,d\theta$$

$$=\frac{1}{2}\int_{-\frac{\pi}{4}}^{\frac{\pi}{4}}\left|2(r^2+a^2)^{\frac{1}{2}}\right|_0^{a\sqrt{\cos 2\theta}} d\theta \qquad \left[\because \int [f(r)]^n f'(r)dr=\frac{[f(r)]^{n+1}}{n+1}\right]$$

$$=\frac{1}{2}\int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} 2a\left[(\cos 2\theta+1)^{\frac{1}{2}}-1\right]d\theta$$

$$=a\int_{-\frac{\pi}{4}}^{\frac{\pi}{4}}\left(\sqrt{2}\cos\theta-1\right)d\theta$$

$$=a\left|\sqrt{2}\sin\theta-\theta\right|_{-\frac{\pi}{4}}^{\frac{\pi}{4}}$$

$$=a\left[\sqrt{2}\sin\frac{\pi}{4}-\frac{\pi}{4}-\sqrt{2}\sin\left(-\frac{\pi}{4}\right)+\left(-\frac{\pi}{4}\right)\right]$$

$$=a\left(2-\frac{\pi}{2}\right)$$

$$=\frac{a}{2}(4-\pi)$$

Example 5

Evaluate $\iint r^2 dr\, d\theta$, *over the area between the circles* $r = a \sin\theta$ *and* $r = 2a \sin\theta$.

Solution

1. The region of integration is the area bounded between the circle $r = a \sin\theta$ and $r = 2a\sin\theta$.
2. Draw an elementary radius vector OAB from the origin which enters in the region from the circle $r = a\sin\theta$ and leaves at the circle $r = 2a\sin\theta$.
3. Limits of r : $r = a\sin\theta$ to $r = 2a\sin\theta$
   Limits of θ : $\theta = 0$ to $\theta = \pi$

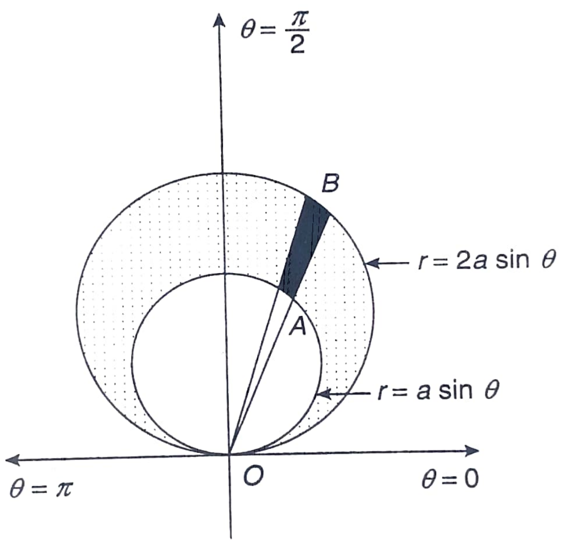

Fig. 5.51

$$I = \iint r^2 dr\, d\theta$$

$$= \int_0^{\pi} \int_{a\sin\theta}^{2a\sin\theta} r^2 dr\, d\theta$$

$$= \int_0^{\pi} \left| \frac{r^3}{3} \right|_{a\sin\theta}^{2a\sin\theta} d\theta$$

$$= \frac{1}{3}\int_0^{\pi} (8a^3 \sin^3\theta - a^3\sin^3\theta) d\theta$$

$$= \frac{7a^3}{3}\int_0^{\pi} \sin^3\theta\, d\theta$$

$$= \frac{7a^3}{3}\int_0^{\pi} \frac{3\sin\theta - \sin 3\theta}{4} d\theta$$

[$\sin 3\theta = 3\sin\theta - 4\sin^3\theta$
$\Rightarrow 4\sin^3\theta = 3\sin\theta - \sin 3\theta$
$\Rightarrow \sin^3\theta = \frac{3\sin\theta - \sin 3\theta}{4}$]

$$= \frac{7a^3}{12}\left|-3\cos\theta + \frac{\cos 3\theta}{3}\right|_0^\pi$$

$$= \frac{7a^3}{12}\left[-3(\cos\pi - \cos 0) + \frac{1}{3}(\cos 3\pi - \cos 0)\right]$$

$$= \frac{7a^3}{12}\left(\frac{16}{3}\right)$$

$$= \frac{28}{9}a^3$$

EXERCISE 5.4

Evaluate the following:

1. $\iint re^{\frac{-r^2}{a^2}}\cos\theta\sin\theta\,dr\,d\theta$, over the upper half of the circle $r = 2a\cos\theta$.

$$\left[\text{Ans.}: \frac{a^2}{16}\left(3 + \frac{1}{e^4}\right)\right]$$

2. $\iint r^3 dr\,d\theta$, over the region between the circles $r = 2\sin\theta$ and $r = 4\sin\theta$.

$$\left[\text{Ans.}: \frac{45\pi}{2}\right]$$

3. $\iint r\sin\theta\,dA$, over the cardioid $r = a(1+\cos\theta)$ above the initial line.

$$\left[\text{Ans.}: \frac{4}{3}a^3\right]$$

4. $\iint \frac{r}{\sqrt{r^2+4}}dr\,d\theta$, over one loop of the lemniscate $r^2 = 4\cos 2\theta$.

$$\left[\text{Ans.}: (4 - \pi)\right]$$

5.5 CHANGE OF VARIABLES IN DOUBLE INTEGRALS

The double integral can be changed from Cartesian coordinates (x, y) to polar coordinates (r, θ) by putting $x = r\cos\theta$, $y = r\sin\theta$. Then $\iint f(x, y)dy\,dx = \iint f(r\cos\theta, r\sin\theta)|J|dr\,d\theta$ where J is the Jacobian (functional determinant) defined as

$$J = \frac{\partial(x, y)}{\partial(r, \theta)} = \begin{vmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta} \end{vmatrix}$$

$$= \begin{vmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{vmatrix}$$

$$= r(\cos^2\theta + \sin^2\theta)$$

$$= r$$

Hence, $\iint f(x, y)\mathrm{d}y\,\mathrm{d}x = \iint f(r\cos\theta, r\sin\theta)|r|\mathrm{d}r\,\mathrm{d}\theta$

$$= \iint f(r\cos\theta, r\sin\theta) r\,\mathrm{d}r\,\mathrm{d}\theta$$

Example 1

Evaluate $\iint \sqrt{\frac{1-x^2-y^2}{1+x^2+y^2}}\,\mathrm{d}x\,\mathrm{d}y$ *over the first quadrant of the circle* $x^2 + y^2 = 1$.

Solution

1. Putting $x = r\cos\theta$, $y = r\sin\theta$, polar form of the circle $x^2 + y^2 = 1$ is obtained as $r = 1$.
2. The region of integration is the part of the circle $r = 1$ in the first quadrant.
3. Draw an elementary radius vector OA which starts from the origin and terminates on the circle $r = 1$.
4. Limits of r : $r = 0$ to $r = 1$

   Limits of θ : $\theta = 0$ to $\theta = \frac{\pi}{2}$

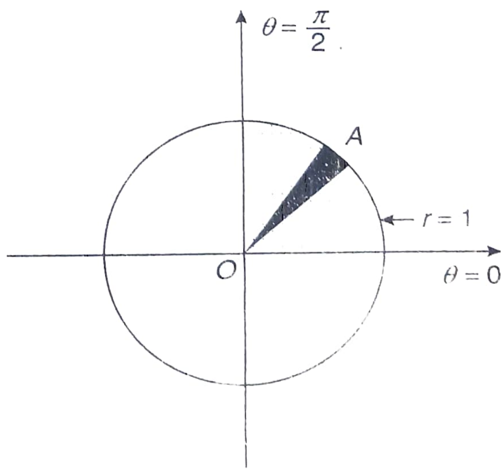

Fig. 5.52

Hence, the polar form of the given integral is

$$I = \iint \sqrt{\frac{1-x^2-y^2}{1+x^2+y^2}}\,\mathrm{d}x\,\mathrm{d}y$$

$$= \int_0^{\frac{\pi}{2}}\int_0^1 \sqrt{\frac{1-r^2}{1+r^2}}\, r\,\mathrm{d}r\,\mathrm{d}\theta$$

Putting $r^2 = \cos 2t$, $2r\,\mathrm{d}r = -2\sin 2t\,\mathrm{d}t$

When $r = 0, t = \frac{\pi}{4}$

When $r = 1, t = 0$

$$I = \int_0^{\frac{\pi}{2}}\int_{\frac{\pi}{4}}^0 \sqrt{\frac{1-\cos 2t}{1+\cos 2t}}(-\sin 2t\,\mathrm{d}t)\mathrm{d}\theta$$

$$= \int_0^{\frac{\pi}{2}}\int_0^{\frac{\pi}{4}} \sqrt{\frac{2\sin^2 t}{2\cos^2 t}}\sin 2t\,\mathrm{d}t\,\mathrm{d}\theta$$

$$= \int_0^{\frac{\pi}{2}} \int_0^{\frac{\pi}{4}} \frac{\sin t}{\cos t} \cdot 2 \sin t \cos t \, dt \, d\theta$$

$$= \int_0^{\frac{\pi}{2}} d\theta \int_0^{\frac{\pi}{4}} (1 - \cos 2t) dt$$

$$= |\theta|_0^{\frac{\pi}{2}} \left| t - \frac{\sin 2t}{2} \right|_0^{\frac{\pi}{4}} d\theta$$

$$= \frac{\pi}{2}\left(\frac{\pi}{4} - \frac{1}{2}\right)$$

$$= \frac{\pi}{8}(\pi - 2)$$

Example 2

Evaluate $\iint \frac{4xy}{x^2 + y^2} e^{-x^2 - y^2} \, dx \, dy$, *over the region bounded by the circle* $x^2 + y^2 - x = 0$ *in the first quadrant.*

Solution

1. Putting $x = r\cos\theta$, $y = r\sin\theta$, polar form of the circle $x^2 + y^2 - x = 0$ is $r^2 - r\cos\theta = 0$, $r = \cos\theta$.
2. The region of integration is the part of the circle $r = \cos\theta$ in the first quadrant.
3. Draw an elementary radius vector OA which starts from the origin and terminates on the circle $r = \cos\theta$.
4. Limits of r : $r = 0$ to $r = \cos\theta$

   Limits of θ : $\theta = 0$ to $\theta = \frac{\pi}{2}$

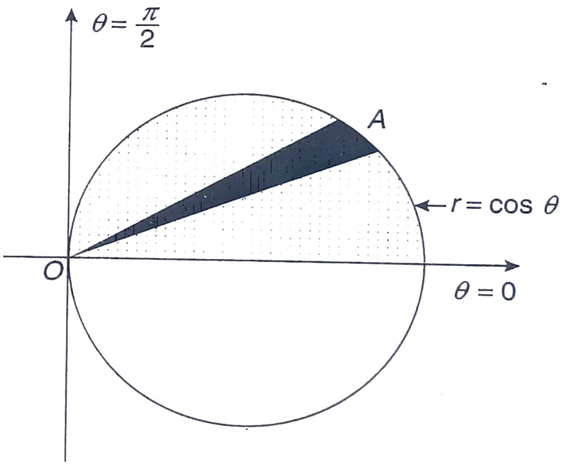

Fig. 5.53

Hence, the polar form of the given integral is

$$I = \iint \frac{4xy}{x^2 + y^2} e^{-x^2 - y^2} \, dx \, dy$$

$$= \int_0^{\frac{\pi}{2}} \int_0^{\cos\theta} \frac{4r^2 \cos\theta \sin\theta}{r^2} e^{-r^2} r \, dr \, d\theta$$

$$= -2\int_0^{\frac{\pi}{2}} \cos\theta \sin\theta \left[\int_0^{\cos\theta} e^{-r^2} (-2r) \, dr \right] d\theta$$

$$= -2\int_0^{\frac{\pi}{2}} \cos\theta \sin\theta \left| e^{-r^2} \right|_0^{\cos\theta} d\theta \qquad \left[\because \int e^{f(r)} f'(r) dr = e^{f(r)} \right]$$

$$= -2\int_0^{\frac{\pi}{2}} \cos\theta \sin\theta \left(e^{-\cos^2\theta} - 1 \right) d\theta$$

$$= -\int_0^{\frac{\pi}{2}} \left[e^{-\cos^2\theta} (2\cos\theta \sin\theta) - \sin 2\theta \right] d\theta$$

$$= -\left| e^{-\cos^2\theta} + \frac{\cos 2\theta}{2} \right|_0^{\frac{\pi}{2}}$$

$$= -\left(e^0 + \frac{\cos\pi}{2} - e^{-1} - \frac{\cos 0}{2} \right)$$

$$= \frac{1}{e}$$

Example 3

Evaluate $\iint \frac{x^2y^2}{(x^2+y^2)}\,dx\,dy$, *over the region bounded by the circles* $x^2+y^2=a^2$ *and* $x^2+y^2=b^2$ $(a>b)$.

Solution

1. Putting $x = r\cos\theta$, $y = r\sin\theta$, polar form of
   (i) the circle $x^2+y^2=a^2$ is $r^2=a^2$, $r=a$.
   (ii) the circle $x^2+y^2=b^2$ is $r^2=b^2$, $r=b$.
2. The region of integration is the part bounded between the circles $r = a$ and $r = b$.
3. Draw an elementary radius vector OAB from the origin which enters in the region from the circle $r = b$ and terminates on the circle $r = a$.
4. Limits of r : $r = b$ to $r = a$
   Limits of θ : $\theta = 0$ to $\theta = 2\pi$

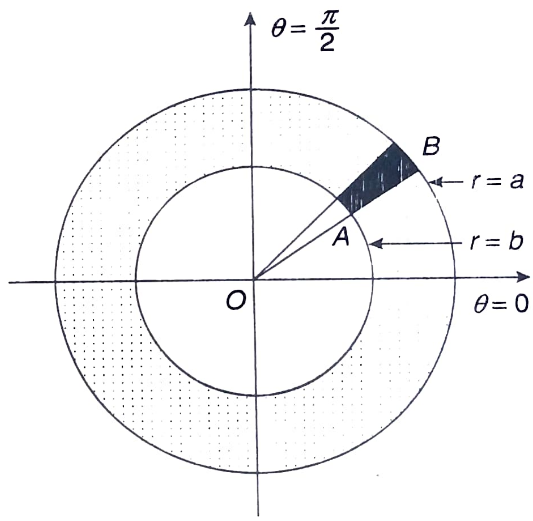

Fig. 5.54

Hence, the polar form of the given integral is

$$I = \iint \frac{x^2y^2}{(x^2+y^2)}\,dx\,dy$$

$$= \int_0^{2\pi}\int_b^a \frac{r^4\cos^2\theta\sin^2\theta}{r^2}\cdot r\,dr\,d\theta$$

$$= \int_0^{2\pi} \cos^2\theta\sin^2\theta \left| \frac{r^4}{4} \right|_b^a d\theta$$

$$= \int_0^{2\pi} \frac{\sin^2 2\theta}{4}\cdot\frac{(a^4-b^4)}{4}\,d\theta$$

$$= \frac{a^4-b^4}{16}\int_0^{2\pi}\frac{(1-\cos 4\theta)}{2}\,d\theta$$

$$= \left(\frac{a^4 - b^4}{32}\right) \left| \theta - \frac{\sin 4\theta}{4} \right|_0^{2\pi}$$

$$= \left(\frac{a^4 - b^4}{32}\right)(2\pi)$$

$$= \frac{\pi}{16}(a^4 - b^4)$$

Example 4

Evaluate $\iint \frac{(x^2+y^2)^2}{x^2y^2} \mathrm{d}x\,\mathrm{d}y$, *over the region common to the circles* $x^2 + y^2 = ax$ *and* $x^2 + y^2 = by$ $(a, b > 0)$.

Solution

1. Putting $x = r\cos\theta$, $y = r\sin\theta$, polar form of
   (i) the circle $x^2 + y^2 = ax$ is $r^2 = ar\cos\theta$, $r = a\cos\theta$.
   (ii) the circle $x^2 + y^2 = by$ is $r^2 = br\sin\theta$, $r = b\sin\theta$.

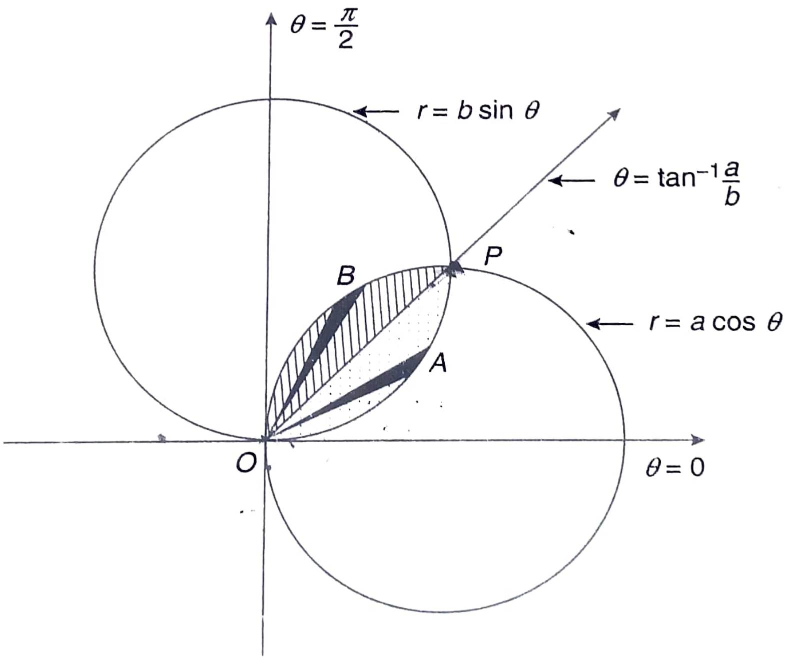

Fig. 5.55

2. The region of integration is the common part of the circles $r = a\cos\theta$ and $r = b\sin\theta$.
3. The point of intersection of the circle $r = a\cos\theta$ and $r = b\sin\theta$, is obtained as

$$b\sin\theta = a\cos\theta$$

$$\tan\theta = \frac{a}{b}$$

$$\theta = \tan^{-1}\frac{a}{b}$$

Hence, $\theta = \tan^{-1}\dfrac{a}{b}$ at P.

4. Divide the region into two subregions OAP and OBP. Draw an elementary radius vector OA and OB in each subregion.
   (i) In subregion OAP, elementary radius vector OA starts from the origin and terminates on the circle $r = b\sin\theta$.
   Limits of r : $r = 0$ to $r = b\sin\theta$
   Limits of θ : $\theta = 0$ to $\theta = \tan^{-1}\dfrac{a}{b}$
   (ii) In subregion OBP, elementary radius vector OB starts from the origin and terminates on the circle $r = a\cos\theta$.
   Limits of r : $r = 0$ to $r = a\cos\theta$
   Limits of θ : $\theta = \tan^{-1}\dfrac{a}{b}$ to $\theta = \dfrac{\pi}{2}$

Hence, the polar form of the given integral is

$$I = \iint \frac{(x^2+y^2)^2}{x^2y^2}\,dx\,dy$$

$$= \int_0^{\tan^{-1}\frac{a}{b}}\int_0^{b\sin\theta} \frac{r^4}{r^4\sin^2\theta\cos^2\theta}\cdot r\,dr\,d\theta + \int_{\tan^{-1}\frac{a}{b}}^{\frac{\pi}{2}}\int_0^{a\cos\theta} \frac{r^4}{r^4\sin^2\theta\cos^2\theta}\cdot r\,dr\,d\theta$$

$$= \int_0^{\tan^{-1}\frac{a}{b}} \frac{1}{\sin^2\theta\cos^2\theta}\left|\frac{r^2}{2}\right|_0^{b\sin\theta} d\theta + \int_{\tan^{-1}\frac{a}{b}}^{\frac{\pi}{2}} \frac{1}{\sin^2\theta\cos^2\theta}\left|\frac{r^2}{2}\right|_0^{a\cos\theta} d\theta$$

$$= \frac{1}{2}\int_0^{\tan^{-1}\frac{a}{b}} \frac{1}{\sin^2\theta\cos^2\theta}\cdot b^2\sin^2\theta\,d\theta + \frac{1}{2}\int_{\tan^{-1}\frac{a}{b}}^{\frac{\pi}{2}} \frac{1}{\sin^2\theta\cos^2\theta}\cdot a^2\cos^2\theta\,d\theta$$

$$= \frac{b^2}{2}\int_0^{\tan^{-1}\frac{a}{b}} \sec^2\theta\,d\theta + \frac{a^2}{2}\int_{\tan^{-1}\frac{a}{b}}^{\frac{\pi}{2}} \operatorname{cosec}^2\theta\,d\theta$$

$$= \frac{b^2}{2}\left|\tan\theta\right|_0^{\tan^{-1}\frac{a}{b}} + \frac{a^2}{2}\left|-\cot\theta\right|_{\tan^{-1}\frac{a}{b}}^{\frac{\pi}{2}}$$

$$= \frac{b^2}{2}\left[\tan\tan^{-1}\left(\frac{a}{b}\right) - \tan 0\right] - \frac{a^2}{2}\left[\cot\frac{\pi}{2} - \cot\left(\tan^{-1}\frac{a}{b}\right)\right]$$

$$= \frac{b^2}{2}\left[\frac{a}{b} - 0\right] - \frac{a^2}{2}\left[0 - \frac{b}{a}\right]$$

$$= \frac{ab}{b} + \frac{ab}{2}$$

$$= ab$$

Example 5

Evaluate $\int_0^\infty \int_0^\infty e^{-(x^2+y^2)} \mathrm{d}x\, \mathrm{d}y.$

Solution

1. Limits of $x : x = 0$ to $x \to \infty$
   Limits of $y : y = 0$ to $y \to \infty$
2. The region of integration is the first quadrant.
3. Putting $x = r\cos\theta$, $y = r\sin\theta$, the integral changes to polar form.
4. Draw an elementary radius vector which starts from the origin and extends up to infinity.
   Limits of r : $r = 0$ to $r \to \infty$
   Limits of θ : $\theta = 0$ to $\theta = \frac{\pi}{2}$

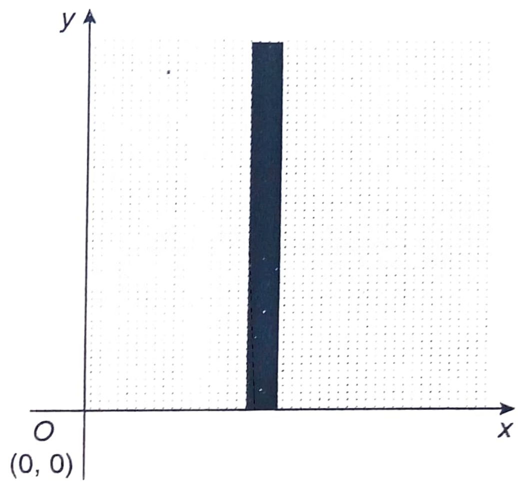

Fig. 5.56(a)

Hence, the polar form of the given integral is

$$I = \int_0^\infty \int_0^\infty e^{-(x^2+y^2)} \mathrm{d}x\, \mathrm{d}y$$

$$= \int_0^{\frac{\pi}{2}} \int_0^\infty e^{-r^2} r\, \mathrm{d}r\, \mathrm{d}\theta$$

$$= -\frac{1}{2} \int_0^{\frac{\pi}{2}} \int_0^\infty e^{-r^2} (-2r) \mathrm{d}r\, \mathrm{d}\theta$$

$$= -\frac{1}{2} \int_0^{\frac{\pi}{2}} \left| e^{-r^2} \right|_0^\infty \mathrm{d}\theta \quad \left[\because \int e^{f(r)} f'(r) \mathrm{d}r = e^{f(r)}\right]$$

$$= -\frac{1}{2} \int_0^{\frac{\pi}{2}} (0 - e^0) \mathrm{d}\theta$$

$$= -\frac{1}{2} \left| -\theta \right|_0^{\frac{\pi}{2}}$$

$$= \frac{\pi}{4}$$

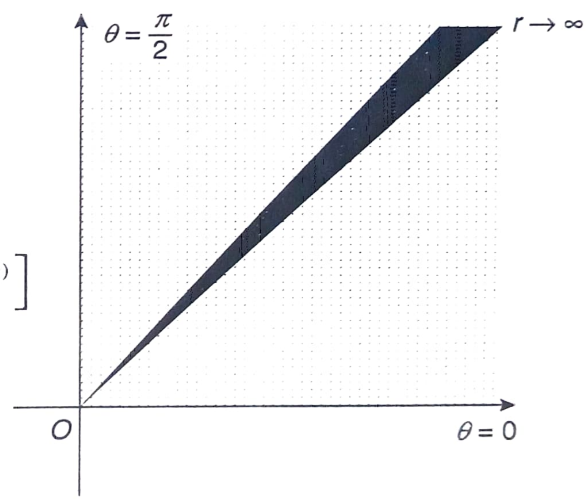

Fig. 5.56(b)

Example 6

Evaluate $\int_{-\infty}^\infty \int_{-\infty}^\infty \frac{\mathrm{d}x\, \mathrm{d}y}{(1 + x^2 + y^2)^{\frac{3}{2}}}.$

Solution

1. Limits of $x : x \to -\infty$ to $x \to \infty$
   Limits of $y : y \to -\infty$ to $y \to \infty$

2. The region of integration is the entire coordinate plane.
3. Putting $x = r\cos\theta$, $y = r\sin\theta$, integral changes to polar form.
4. Draw an elementary radius vector which starts from origin and extends up to ∞.
   Limits of r : $r = 0$ to $r \to \infty$
   Limits of θ : $\theta = 0$ to $\theta = 2\pi$

Hence, the polar form of the given integral is

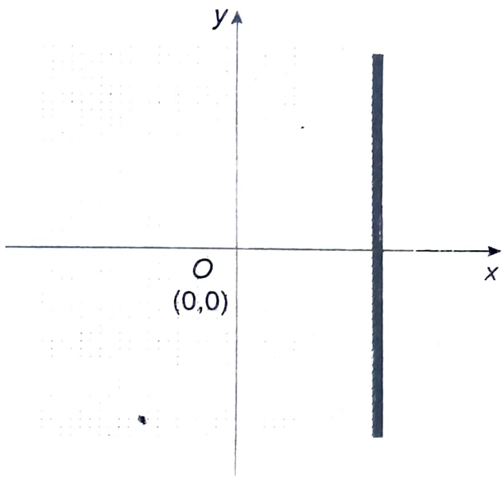

Fig. 5.57(a)

$$I = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \frac{dx\,dy}{(1+x^2+y^2)^{\frac{3}{2}}}$$

$$= \int_0^{2\pi}\int_0^{\infty} \frac{r\,dr\,d\theta}{(1+r^2)^{\frac{3}{2}}}$$

$$= \frac{1}{2}\int_0^{2\pi}\int_0^{\infty} (1+r^2)^{-\frac{3}{2}}(2r)dr\,d\theta$$

$$= \frac{1}{2}\int_0^{2\pi} \left| -2(1+r^2)^{-\frac{1}{2}} \right|_0^{\infty} d\theta$$

$$\left[\because \int [f(r)]^n f'(r)dr = \frac{[f(r)]^{n+1}}{n+1}\right]$$

$$= -\int_0^{2\pi} \left| \frac{1}{\sqrt{1+r^2}} \right|_0^{\infty} d\theta$$

$$= -\int_0^{2\pi} (0-1)d\theta$$

$$= |\theta|_0^{2\pi}$$

$$= 2\pi$$

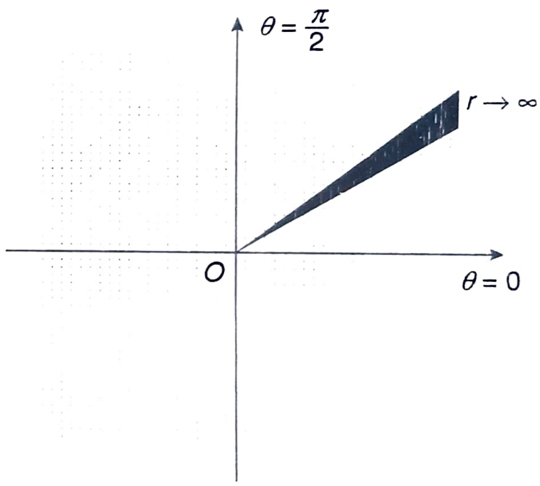

Fig. 5.57(b)

Example 7

Evaluate $\int_0^a \int_y^a \frac{x}{x^2+y^2}\,dx\,dy$ *by transforming into polar coordinates.*

Solution

1. Limits of x : $x = y$ to $x = a$
   Limits of y : $y = 0$ to $y = a$
2. The region of integration is bounded by the lines $y = x$, $x = a$ and $y = 0$.
3. Putting $x = r\cos\theta$, $y = r\sin\theta$, polar form of

   (i) the line $y = x$ is $r\sin\theta = r\cos\theta$, $\tan\theta = 1$, $\theta = \frac{\pi}{4}$.

(ii) the line $x = a$ is $r \cos \theta = a$,

$r = a \sec \theta.$

4. Draw an elementary radius vector OA which starts from the origin and terminates on the line $r = a \sec \theta$.

Limits of $r : r = 0$ to $r = a \sec \theta$

Limits of $\theta : \theta = 0$ to $\theta = \dfrac{\pi}{4}$

Hence, the polar form of the given integral is

$$I = \int_0^a \int_y^a \frac{x}{x^2 + y^2}\, dx\, dy$$

$$= \int_0^{\frac{\pi}{4}} \int_0^{a\sec\theta} \frac{r\cos\theta}{r^2} \cdot r\, dr\, d\theta$$

$$= \int_0^{\frac{\pi}{4}} \int_0^{a\sec\theta} \cos\theta\, dr\, d\theta$$

$$= \int_0^{\frac{\pi}{4}} \left| r \right|_0^{a\sec\theta} \cos\theta\, d\theta$$

$$= \int_0^{\frac{\pi}{4}} a\sec\theta\cos\theta\, d\theta$$

$$= a\int_0^{\frac{\pi}{4}} d\theta$$

$$= a\left|\theta\right|_0^{\frac{\pi}{4}}$$

$$= a \cdot \frac{\pi}{4}$$

$$= \frac{\pi a}{4}$$

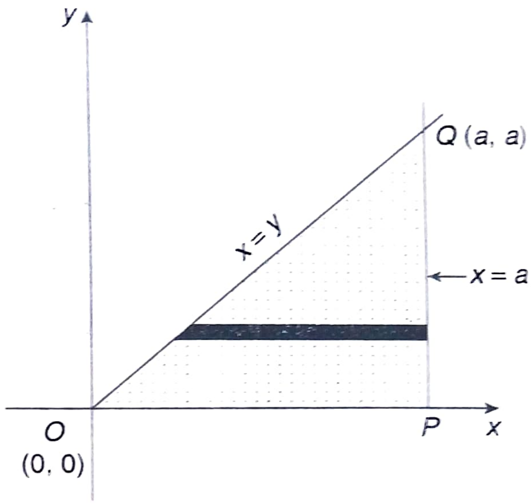

Fig. 5.58(a)

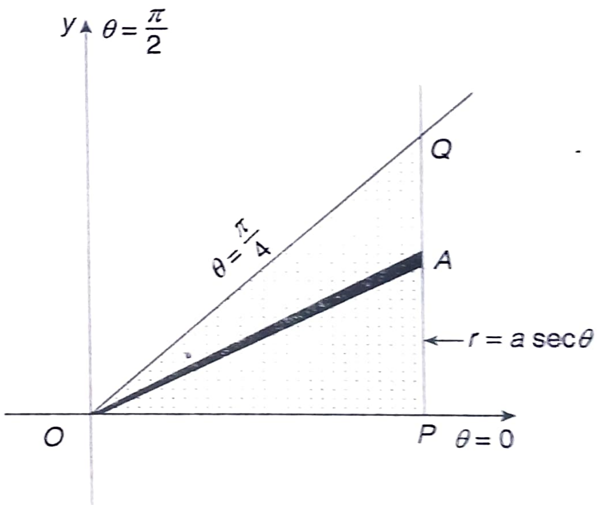

Fig. 5.58(b)

Example 8

Evaluate $\displaystyle\int_0^2 \int_0^{\sqrt{2x-x^2}} \frac{x}{x^2 + y^2}\, dy\, dx$ *by transforming into polar coordinates.*

Solution

1. Limits of $y : y = 0$ to $y = \sqrt{2x - x^2}$

   Limits of $x : x = 0$ to $x = 2$
2. The region of integration is bounded by the circle $x^2 + y^2 - 2x = 0$ and the lines $y = 0$, $x = 0$. Since the limits of x and y are positive, the region of integration is the part of the circle in the first quadrant.
3. Putting $x = r \cos \theta$, $y = r \sin \theta$, polar form of the circle $x^2 + y^2 - 2x = 0$ is

$$r^2 - 2r\cos\theta = 0$$
$$r = 2\cos\theta.$$

4. Draw an elementary radius vector OA which starts from the origin and terminates on the circle $r = 2\cos\theta$.

   Limits of r : $r = 0$ to $r = 2\cos\theta$

   Limits of θ : $\theta = 0$ to $\theta = \frac{\pi}{2}$

Hence, the polar form of the given integral is

$$\begin{aligned} I &= \int_0^2 \int_0^{\sqrt{2x-x^2}} \frac{x}{x^2+y^2}\, dy\, dx \\ &= \int_0^{\frac{\pi}{2}} \int_0^{2\cos\theta} \frac{r\cos\theta}{r^2} \cdot r\, dr\, d\theta \\ &= \int_0^{\frac{\pi}{2}} \int_0^{2\cos\theta} \cos\theta\, dr\, d\theta \\ &= \int_0^{\frac{\pi}{2}} \left| r \right|_0^{2\cos\theta} \cos\theta\, d\theta \\ &= \int_0^{\frac{\pi}{2}} 2\cos^2\theta\, d\theta \\ &= \int_0^{\frac{\pi}{2}} (1+\cos 2\theta)\, d\theta \\ &= \left| \theta + \frac{\sin 2\theta}{2} \right|_0^{\frac{\pi}{2}} \\ &= \frac{\pi}{2} + \frac{1}{2}\sin\pi - \frac{1}{2}\sin 0 \\ &= \frac{\pi}{2} \end{aligned}$$

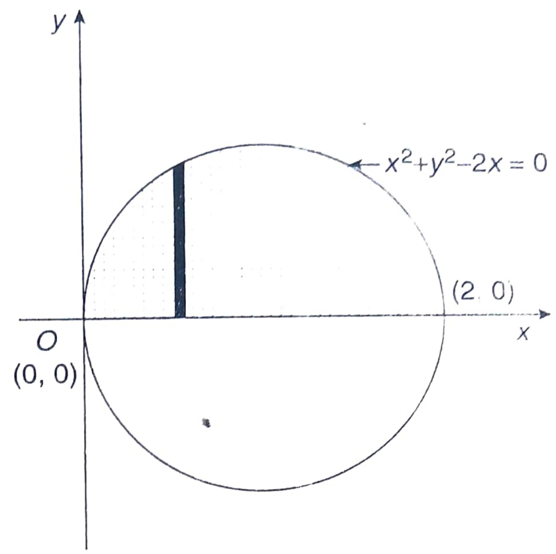

Fig. 5.59(a)

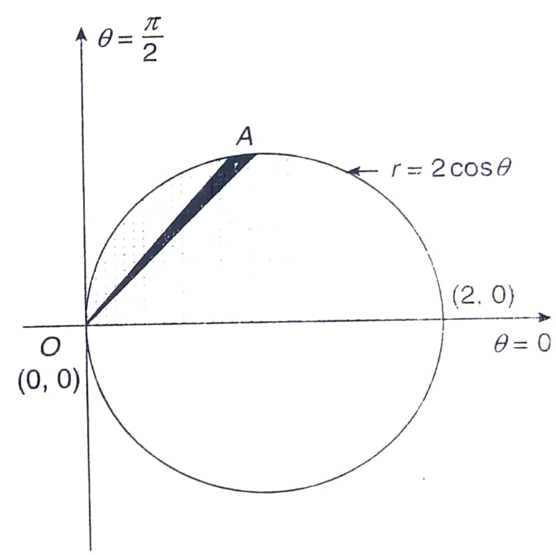

Fig. 5.59(b)

Example 9

Evaluate $\int_0^1 \int_x^{\sqrt{2x-x^2}} (x^2+y^2)\, dx\, dy.$

Solution

1. Limits of y : $y = x$ to $y = \sqrt{2x-x^2}$

   Limits of x : $x = 0$ to $x = 1$
2. The region of integration is bounded by the line $y = x$ and the circle $x^2 + y^2 - 2x = 0$.
3. Putting $x = r\cos\theta$, $y = r\sin\theta$, polar form of

(i) the line $y = x$ is

$$r\sin\theta = r\cos\theta,\ \tan\theta = 1,\ \theta = \frac{\pi}{4}.$$

(ii) the circle $x^2 + y^2 - 2x = 0$ is

$$r^2 - 2r\cos\theta = 0$$

$$r = 2\cos\theta.$$

4. Draw an elementary radius vector OA which starts from the origin and terminates on the circle $r = 2\cos\theta$.

Limits of r : $r = 0$ to $r = 2\cos\theta$

Limits of $\theta : \theta = \frac{\pi}{4}$ to $\theta = \frac{\pi}{2}$

Hence, the polar form of the given integral is

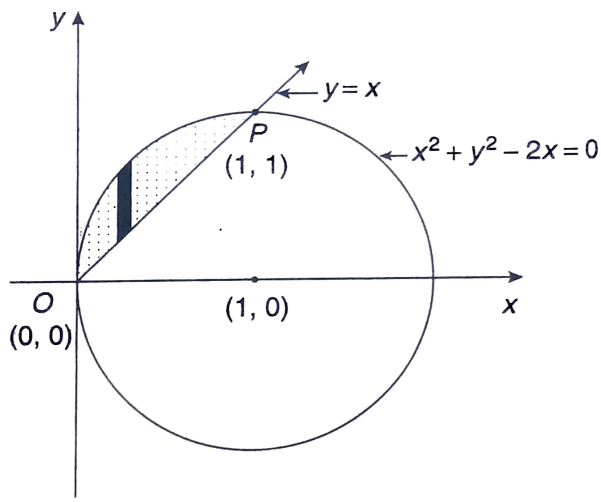

Fig. 5.60(a)

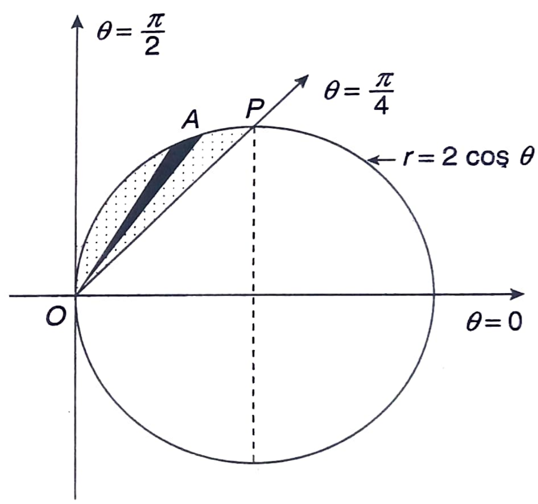

Fig. 5.60(b)

$$I = \int_0^1 \int_x^{\sqrt{2x-x^2}} (x^2 + y^2)\,dx\,dy$$

$$= \int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \int_0^{2\cos\theta} r^2 \cdot r\,dr\,d\theta$$

$$= \int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \left|\frac{r^4}{4}\right|_0^{2\cos\theta} d\theta$$

$$= 4\int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \cos^4\theta\,d\theta$$

$$= 4\int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \left(\frac{1+\cos 2\theta}{2}\right)^2 d\theta$$

$$= \int_{\frac{\pi}{4}}^{\frac{\pi}{2}} (1 + 2\cos 2\theta + \cos^2 2\theta)\,d\theta$$

$$= \int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \left(1 + 2\cos 2\theta + \frac{1+\cos 4\theta}{2}\right) d\theta$$

$$= \left|\frac{3}{2}\theta + \frac{2\sin 2\theta}{2} + \frac{\sin 4\theta}{8}\right|_{\frac{\pi}{4}}^{\frac{\pi}{2}}$$

$$= \frac{3}{2}\left(\frac{\pi}{2} - \frac{\pi}{4}\right) - \left(\sin\pi - \sin\frac{\pi}{2}\right) + \frac{1}{8}(\sin 2\pi - \sin\pi)$$

$$= \frac{3\pi}{8} + 1$$

Example 10

Evaluate $\displaystyle\int_0^1 \int_{\sqrt{x-x^2}}^{\sqrt{1-x^2}} \frac{xye^{-(x^2+y^2)}}{x^2+y^2}\,dx\,dy.$

Solution

1. Limits of $y: y=\sqrt{x-x^2}$ to $y=\sqrt{1-x^2}$
   Limits of $x: x=0$ to $x=1$
2. The region of integration is the part of the first quadrant bounded by the circles $x^2+y^2-x=0$ and $x^2+y^2=1$.
3. Putting $x = r\cos\theta$, $y = r\sin\theta$, polar form of
   (i) the circle $x^2+y^2-x=0$ is $r^2-r\cos\theta=0$, $r=\cos\theta$.
   (ii) the circle $x^2+y^2=1$ is $r^2=1$, $r=1$.
4. Draw an elementary radius vector OAB from the origin which enters in the region from the circle $r=\cos\theta$ and terminates on the circle $r=1$.
   Limits of r : $r=\cos\theta$ to $r=1$
   Limits of θ: $\theta=0$ to $\theta=\frac{\pi}{2}$

Hence, the polar form of the given integral is

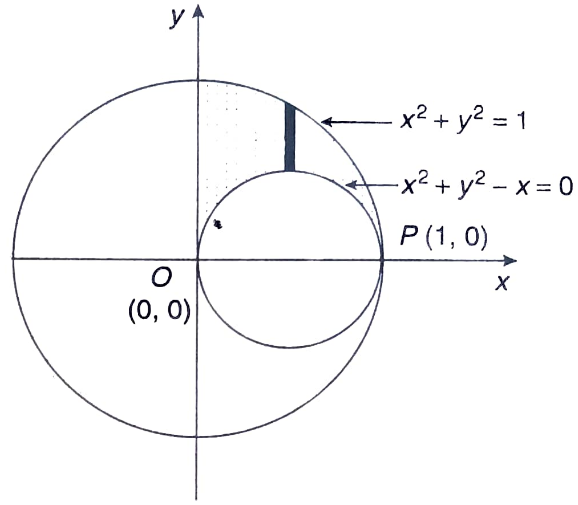

Fig. 5.61(a)

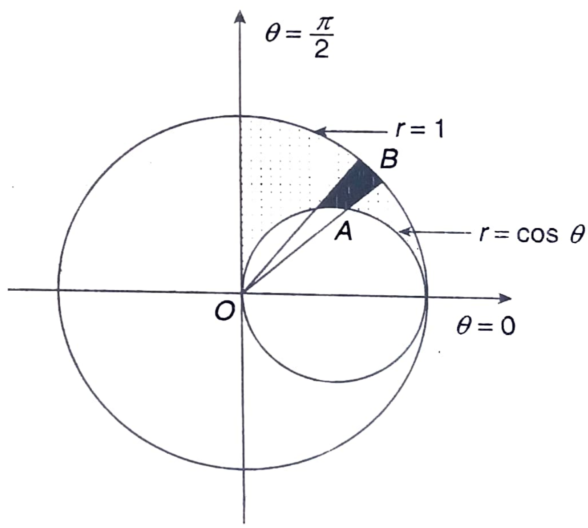

Fig. 5.61(b)

$$I=\int_0^1\int_{\sqrt{x-x^2}}^{\sqrt{1-x^2}}\frac{xye^{-(x^2+y^2)}}{x^2+y^2}\,dx\,dy$$

$$=\int_0^{\frac{\pi}{2}}\int_{\cos\theta}^{1}\frac{r^2\sin\theta\cos\theta e^{-r^2}}{r^2}\cdot r\,dr\,d\theta$$

$$=-\frac{1}{2}\int_0^{\frac{\pi}{2}}\sin\theta\cos\theta\int_{\cos\theta}^{1}e^{-r^2}(-2r)\,dr\,d\theta$$

$$=-\frac{1}{2}\int_0^{\frac{\pi}{2}}\sin\theta\cos\theta\left|e^{-r^2}\right|_{\cos\theta}^{1}d\theta$$

$$\left[\because\int e^{f(r)}f'(r)dr=e^{f(r)}\right]$$

$$=-\frac{1}{2}\int_0^{\frac{\pi}{2}}\sin\theta\cos\theta(e^{-1}-e^{-\cos^2\theta})d\theta$$

$$=-\frac{1}{2}\int_0^{\frac{\pi}{2}}\frac{1}{2}\left(\frac{1}{e}\sin 2\theta-e^{-\cos^2\theta}\cdot 2\sin\theta\cos\theta\right)d\theta$$

$$=-\frac{1}{4}\left|\frac{1}{e}\left(-\frac{\cos 2\theta}{2}\right)-e^{-\cos^2\theta}\right|_0^{\frac{\pi}{2}}\quad\left[\because\int e^{f(\theta)}f'(\theta)d\theta=e^{f(\theta)}\right]$$

$$=-\frac{1}{4}\left[-\frac{1}{2e}(\cos\pi-\cos 0)-e^{-\left(\cos^2\frac{\pi}{2}\right)}+e^{-\cos^2 0}\right]$$

$$=-\frac{1}{4}\left[-\frac{1}{2e}(-2)-e^{0}+e^{-1}\right]$$

$$= -\frac{1}{4}\left[\frac{1}{e} - 1 + \frac{1}{e}\right]$$

$$= \frac{1}{4}\left[1 - \frac{2}{e}\right]$$

Example 11

Evaluate $\int_0^{4a}\int_{\frac{y^2}{4a}}^{y} \frac{x^2 - y^2}{x^2 + y^2}\,dx\,dy.$

Solution

1. Limits of $x : x = \frac{y^2}{4a}$ to $x = y$

   Limits of $y : y = 0$ to $y = 4a.$
2. The region of integration is bounded by the line $y = x$ and the parabola $y^2 = 4ax.$
3. Putting $x = r\cos\theta$, $y = r\sin\theta$, polar form of

   (i) the line $y = x$ is $r\sin\theta = r\cos\theta$,

   $\tan\theta = 1,\ \theta = \frac{\pi}{4}.$

   (ii) the parabola $y^2 = 4ax$ is

   $r^2\sin^2\theta = 4ar\cos\theta,\ r = 4a\cot\theta\operatorname{cosec}\theta.$
4. Draw an elementary radius vector OA which starts from the origin and terminates on the parabola $r = 4a\cot\theta\operatorname{cosec}\theta.$

   Limits of $r : r = 0$ to $r = 4a\cot\theta\operatorname{cosec}\theta$

   Limits of $\theta : \theta = \frac{\pi}{4}$ to $\theta = \frac{\pi}{2}$

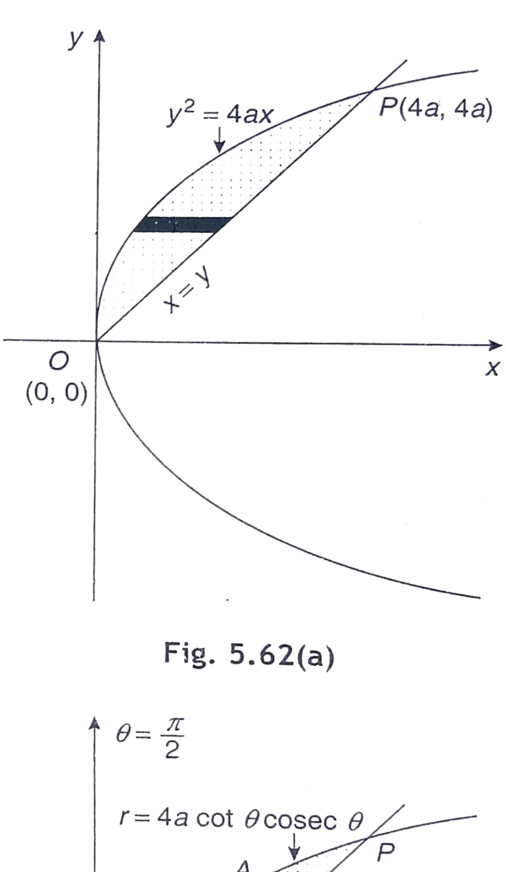

Fig. 5.62(a)

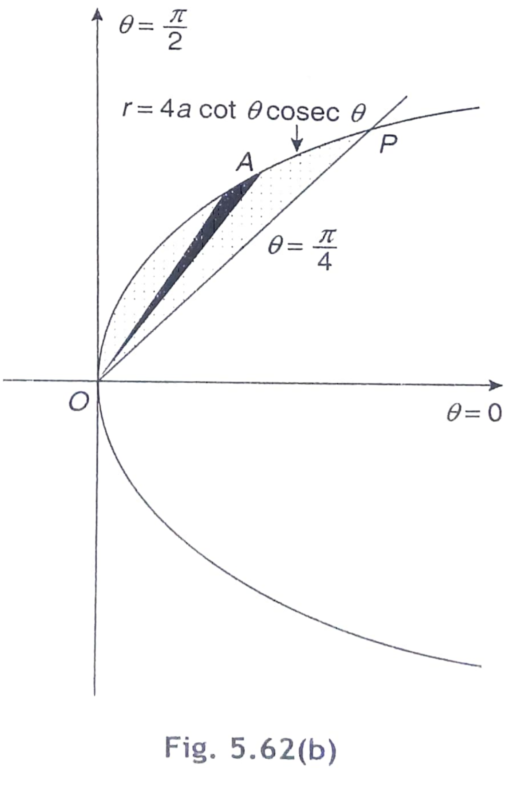

Fig. 5.62(b)

Hence, the polar form of the given integral is

$$I = \int_0^{4a}\int_{\frac{y^2}{4a}}^{y} \frac{x^2 - y^2}{x^2 + y^2}\,dx\,dy$$

$$= \int_{\frac{\pi}{4}}^{\frac{\pi}{2}}\int_0^{4a\cot\theta\operatorname{cosec}\theta} \frac{r^2(\cos^2\theta - \sin^2\theta)}{r^2}\cdot r\,dr\,d\theta$$

$$= \int_{\frac{\pi}{4}}^{\frac{\pi}{2}} (1 - 2\sin^2\theta)\left|\frac{r^2}{2}\right|_0^{4a\cot\theta\operatorname{cosec}\theta} d\theta$$

$$= \frac{1}{2}\int_{\frac{\pi}{4}}^{\frac{\pi}{2}} (1 - 2\sin^2\theta)(4a)^2\cot^2\theta\operatorname{cosec}^2\theta\,d\theta$$

$$= 8a^2\int_{\frac{\pi}{4}}^{\frac{\pi}{2}} (\cot^2\theta\operatorname{cosec}^2\theta - 2\cot^2\theta)\,d\theta$$

$$= 8a^2 \int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \left[\left\{-(\cot^2\theta)(-\text{cosec}^2\theta)\right\} - 2\text{cosec}^2\theta + 2\right] d\theta$$

$$= 8a^2 \left| -\frac{\cot^3\theta}{3} + 2\cot\theta + 2\theta \right|_{\frac{\pi}{4}}^{\frac{\pi}{2}} \qquad \left[\because \int [f(\theta)]^n f'(\theta)d\theta = \frac{[f(\theta)]^{n+1}}{n+1}\right]$$

$$= 8a^2 \left[-\frac{1}{3}\left(\cot^3\frac{\pi}{2} - \cot^3\frac{\pi}{4}\right) + 2\left(\cot\frac{\pi}{2} - \cot\frac{\pi}{4}\right) + 2\left(\frac{\pi}{2} - \frac{\pi}{4}\right)\right]$$

$$= 8a^2 \left[-\frac{1}{3}(-1) + 2(-1) + 2\cdot\frac{\pi}{4}\right]$$

$$= 8a^2 \left[-\frac{5}{3} + \frac{\pi}{2}\right]$$

$$= 8a^2 \left[\frac{\pi}{2} - \frac{5}{3}\right]$$

Example 12

Evaluate $\int_0^a \int_{2\sqrt{ax}}^{\sqrt{5ax-x^2}} \frac{\sqrt{x^2+y^2}}{y^2}\, dx\, dy.$

Solution

1. Limits of $y : y = 2\sqrt{ax}$ to $y = \sqrt{5ax - x^2}$
   Limits of x : $x = 0$ to $x = a$
2. Since the limits of x and y are positive, the region of integration is the part of the first quadrant bounded by the parabola $y^2 = 4ax$ and the circle $x^2 + y^2 - 5ax = 0$
3. Putting $x = r\cos\theta$, $y = r\sin\theta$, polar form of
   (i) the parabola $y^2 = 4ax$ is
   $$r^2\sin^2\theta = 4ar\cos\theta,$$
   $$r = 4a\cot\theta\,\text{cosec}\,\theta.$$
   (ii) the circle $x^2 + y^2 - 5ax = 0$ is
   $$r^2 - 5ar\cos\theta = 0,$$
   $$r = 5a\cos\theta.$$
4. The points of intersection of $r = 4a\cot\theta\,\text{cosec}\,\theta$ and $r = 5a\cos\theta$ are obtained as
   $$4a\cot\theta\,\text{cosec}\,\theta = 5a\cos\theta$$
   $$\sin^2\theta = \frac{4}{5}$$
   $$\theta = \pm\sin^{-1}\frac{2}{\sqrt{5}}$$

Hence, $\theta = \sin^{-1}\frac{2}{\sqrt{5}}$ at P.

5. Draw an elementary radius vector OAB from the origin which enters in the region from the parabola $r = 4a \cot\theta \operatorname{cosec}\theta$ and terminates on the circle $r = 5a \cos\theta$.
Limits of r : $r = 4a \cot\theta \operatorname{cosec}\theta$ to $r = 5a\cos\theta$

Limits of $\theta : \theta = \sin^{-1}\dfrac{2}{\sqrt{5}}$ to $\theta = \dfrac{\pi}{2}$

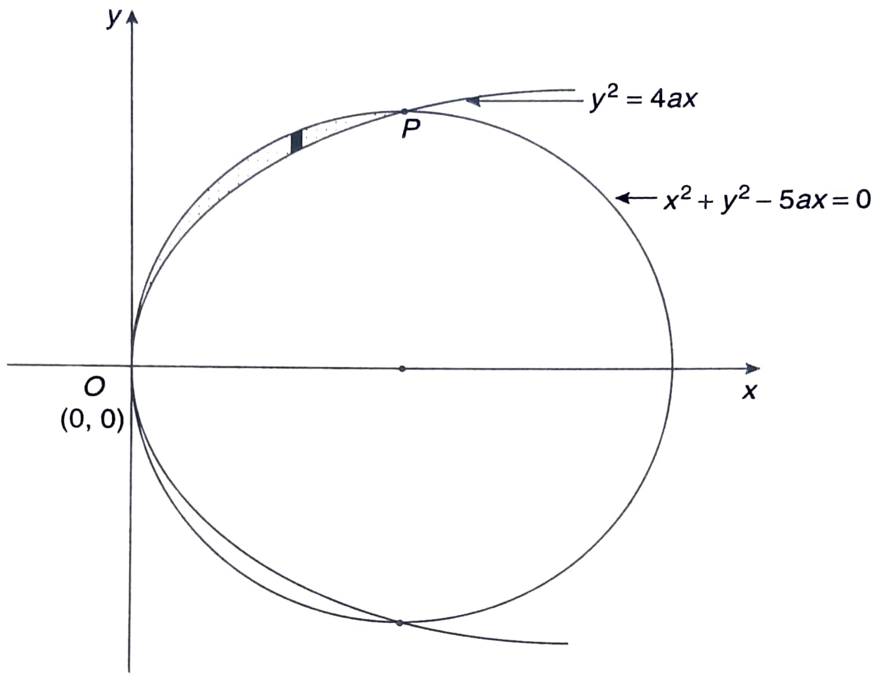

Fig. 5.63(a)

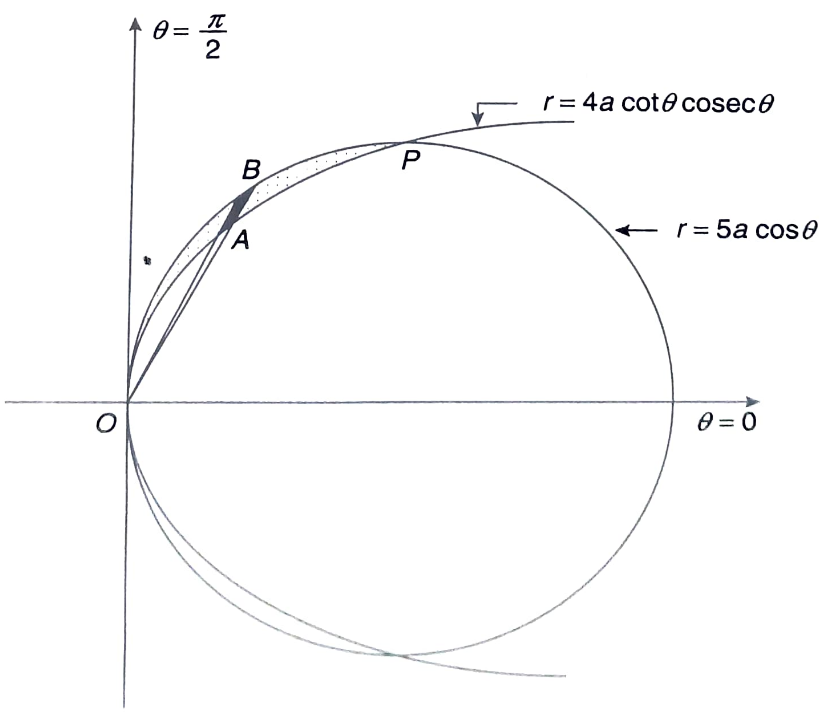

Fig. 5.63(b)

Hence, the polar form of the given integral is

$$I = \int_0^a \int_{2\sqrt{ax}}^{\sqrt{5ax-x^2}} \frac{\sqrt{x^2+y^2}}{y^2}\, dx\, dy$$

$$= \int_{\sin^{-1}\frac{2}{\sqrt{5}}}^{\frac{\pi}{2}} \int_{4a\cot\theta\operatorname{cosec}\theta}^{5a\cos\theta} \frac{r}{r^2\sin^2\theta}\, r\, dr\, d\theta$$

$$= \int_{\sin^{-1}\frac{2}{\sqrt{5}}}^{\frac{\pi}{2}} \operatorname{cosec}^2\theta \left| r \right|_{4a\cot\theta\operatorname{cosec}\theta}^{5a\cos\theta} d\theta$$

$$= \int_{\sin^{-1}\frac{2}{\sqrt{5}}}^{\frac{\pi}{2}} \operatorname{cosec}^2\theta (5a\cos\theta - 4a\cot\theta\operatorname{cosec}\theta) d\theta$$

$$= \int_{\sin^{-1}\frac{2}{\sqrt{5}}}^{\frac{\pi}{2}} \left[5a\cot\theta\operatorname{cosec}\theta + 4a\operatorname{cosec}^2\theta(-\operatorname{cosec}\theta\cot\theta)\right] d\theta$$

$$= \left| -5a\operatorname{cosec}\theta + 4a\frac{\operatorname{cosec}^3\theta}{3} \right|_{\sin^{-1}\frac{2}{\sqrt{5}}}^{\frac{\pi}{2}} \qquad \left[\because \int [f(\theta)]^n f'(\theta) d\theta = \frac{[f(\theta)]^{n+1}}{n+1}\right]$$

$$= \left[-5a\operatorname{cosec}\frac{\pi}{2} + 5a\operatorname{cosec}\left(\sin^{-1}\frac{2}{\sqrt{5}}\right) + \frac{4a}{3}\operatorname{cosec}^3\frac{\pi}{2} - \frac{4a}{3}\operatorname{cosec}^3\left(\sin^{-1}\frac{2}{\sqrt{5}}\right)\right]$$

$$= \left[-5a + 5a\frac{\sqrt{5}}{2} + \frac{4a}{3} - \frac{4a}{3}\left(\frac{\sqrt{5}}{2}\right)^3\right] \qquad \left[\because \operatorname{cosec}\left(\sin^{-1}\frac{2}{\sqrt{5}}\right) = \operatorname{cosec}\left(\operatorname{cosec}^{-1}\frac{\sqrt{5}}{2}\right) = \frac{\sqrt{5}}{2}\right]$$

$$= \frac{a}{3}(5\sqrt{5} - 11)$$

Example 13

Evaluate $\iint xy\left(\frac{x^2}{a^2} + \frac{y^2}{b^2}\right)^{\frac{n}{2}} dx\, dy$, *over the first quadrant of the ellipse* $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$.

Solution

Let $\quad x = ar\cos\theta,\ y = br\cos\theta$

$$J = \frac{\partial(x, y)}{\partial(r, \theta)} = \begin{vmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta} \end{vmatrix}$$

$$= \begin{vmatrix} a\cos\theta & -ar\sin\theta \\ b\sin\theta & br\cos\theta \end{vmatrix} = abr$$

$$dx\, dy = |J|\, dr\, d\theta = abr\, dr\, d\theta$$

Under the transformation $x = ar\cos\theta$, $y = br\sin\theta$, the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ in the xy-plane gets transformed to $r^2 = 1$ or $r = 1$, circle with centre $(0, 0)$ and radius 1 in the $r\theta$-plane.

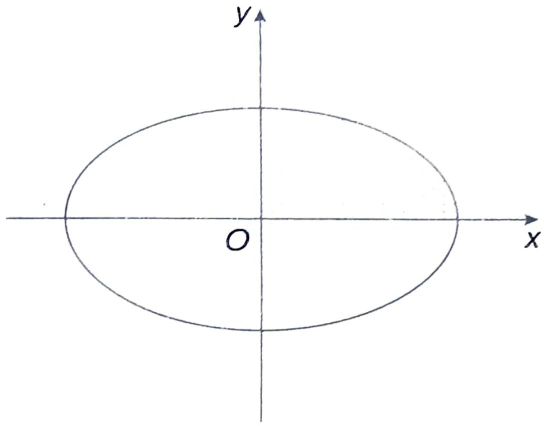

Fig. 5.64

The region of integration is the part of the circle $r = 1$ in first quadrant in the $r\theta$-plane. Draw an elementary radius vector OA which starts from the origin and terminates on the circle $r = 1$.

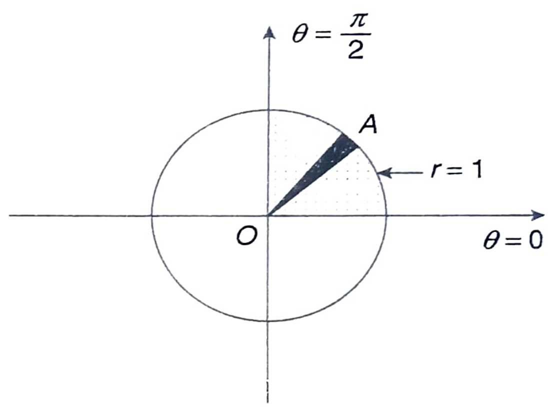

Fig. 5.65

Limits of r : $r = 0$ to $r = 1$

Limits of θ : $\theta = 0$ to $\theta = \frac{\pi}{2}$

Hence, the polar form of the given integral is

$$\mathrm{I} = \iint xy\left(\frac{x^2}{a^2}+\frac{y^2}{b^2}\right)^{\frac{n}{2}} dx\,dy$$

$$= \int_0^{\frac{\pi}{2}}\int_0^1 abr^2\cos\theta\sin\theta(r^2)^{\frac{n}{2}}\,abr\,dr\,d\theta$$

$$= a^2b^2\int_0^{\frac{\pi}{2}}\frac{\sin 2\theta}{2}\int_0^1 (r)^{n+3}\,dr$$

$$= \frac{a^2b^2}{2}\left|-\frac{\cos 2\theta}{2}\right|_0^{\frac{\pi}{2}}\left|\frac{r^{n+4}}{n+4}\right|_0^1$$

$$= \frac{a^2b^2}{4}(-\cos\pi+\cos 0)\cdot\frac{1}{n+4}$$

$$= \frac{a^2b^2}{2(n+4)}$$

EXERCISE 5.5

Change to polar coordinates and evaluate the following:

1. $\iint \frac{1}{\sqrt{xy}}\,dx\,dy$, over the region bounded by the semicircle $x^2 + y^2 - x = 0$, $y \geq 0$.

$$\left[\text{Ans.}: \frac{\pi}{\sqrt{2}}\right]$$

2. $\iint y^2\,dx\,dy$, over the area outside the circle $x^2 + y^2 - ax = 0$ and inside the circle $x^2 + y^2 - 2ax = 0$.

$$\left[\text{Ans.}: \frac{15\pi a^4}{64}\right]$$

3. $\iint \sin(x^2+y^2)\,dx\,dy$, over the circle $x^2+y^2=a^2$.

$$\left[\text{Ans.}: \pi(1-\cos a^2)\right]$$

4. $\iint xy(x^2+y^2)^{\frac{3}{2}}\,dx\,dy$, over the first quadrant of the circle $x^2+y^2=a^2$.

$$\left[\text{Ans.}: \frac{a^7}{14}\right]$$

5. $\int_0^3\int_0^{\sqrt{3x}} \frac{dy\,dx}{\sqrt{x^2+y^2}}$

$$\left[\text{Ans.}: \frac{3}{2}\log 3\right]$$

6. $\int_0^a\int_0^x \frac{x^3\,dx\,dy}{\sqrt{x^2+y^2}}$

$$\left[\text{Ans.}: \frac{a^4}{4}\log\left(1+\sqrt{2}\right)\right]$$

7. $\int_0^a\int_0^{\sqrt{a^2-x^2}} \sin\left[\frac{\pi}{a^2}(a^2-x^2-y^2)\right]dx\,dy$

$$\left[\text{Ans.}: \frac{a^2}{2}\right]$$

8. $\int_0^a\int_0^{\sqrt{a^2-x^2}} e^{-x^2-y^2}\,dx\,dy$

$$\left[\text{Ans.}: \frac{\pi}{4}\left(1-e^{-a^2}\right)\right]$$

9. $\int_0^{2a}\int_0^{\sqrt{2ax-x^2}} (x^2+y^2)\,dx\,dy$

$$\left[\text{Ans.}: \frac{3\pi a^4}{4}\right]$$

10. $\int_0^1\int_0^{\sqrt{x-x^2}} \frac{4xy}{x^2+y^2}e^{-(x^2+y^2)}\,dx\,dy$

$$\left[\text{Ans.}: \frac{1}{e}\right]$$

11. $\int_0^{\frac{a}{\sqrt{2}}}\int_y^{\sqrt{a^2-y^2}} \log e(x^2+y^2)\,dx\,dy$

$$\left[\text{Ans.}: \frac{\pi}{4}a^2\left(\log a-\frac{1}{2}\right)\right]$$

12. $\int_0^a \int_y^{a+\sqrt{a^2-y^2}} \frac{dx\,dy}{(4a^2+x^2+y^2)^3}$

$$\left[\text{Ans.}: \frac{1}{8a^2}\left(\frac{\pi}{4}-\frac{1}{\sqrt{2}}\tan^{-1}\frac{1}{\sqrt{2}}\right)\right]$$

13. $\int_0^a \int_{\sqrt{ax-x^2}}^{\sqrt{a^2-x^2}} \frac{dx\,dy}{\sqrt{a^2-x^2-y^2}}$

$$[\text{Ans.}: a]$$

14. $\int_0^a \int_{\sqrt{ax-x^2}}^{\sqrt{a^2-x^2}} \frac{xy}{x^2+y^2} e^{-(x^2+y^2)} dx\,dy$

$$\left[\text{Ans.}: \frac{1}{4a^2}\left[1-(1+a^2)e^{-a^2}\right]\right]$$

15. $\int_0^1 \int_{x^2}^{x} \frac{dx\,dy}{\sqrt{x^2+y^2}}$

$$\left[\text{Ans.}: \sqrt{2}-1\right]$$