Solution

1. Since inner limits depend on x, the function is integrated first w.r.t. y.
2. Limits of y : $y = x$ to $y = a$, along vertical strip
   Limits of x : $x = 0$ to $x = a$
3. The region is bounded by the lines $y = x$, $y = a$ and $x = 0$
4. The point of intersection of $y = x$ and $y = a$ is $Q\,(a, a)$
5. To change the order of integration, i.e. to integrate first w.r.t. x, draw a horizontal strip AB parallel to x-axis which starts from y-axis and terminates on the line $y = x$.
   Limits of x : $x = 0$ to $x = y$
   Limits of y : $y = 0$ to $y = a$

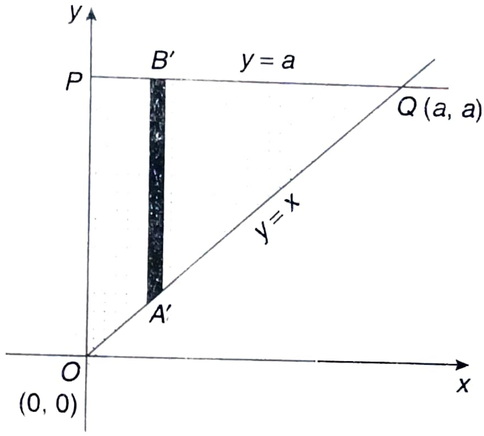

Fig. 5.32(a)

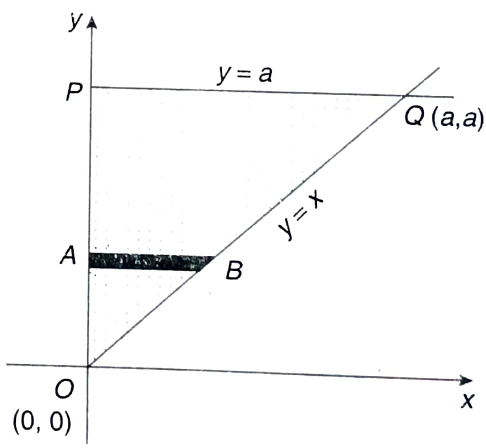

Fig. 5.32(b)

Hence, the given integral after change of order is

$$\int_0^a \int_x^a (x^2 + y^2)\,dy\,dx = \int_0^a \int_0^y (x^2 + y^2)\,dx\,dy$$

$$= \int_0^a \left| \frac{x^3}{3} + y^2 x \right|_0^y dy$$

$$= \int_0^a \left(\frac{y^3}{3} + y^3 \right) dy$$

$$= \int_0^a \frac{4}{3} y^3\,dy$$

$$= \frac{4}{3} \left| \frac{y^4}{4} \right|_0^a$$

$$= \frac{a^4}{3}$$

Example 2

Change the order of integration and evaluate $\int_0^{\pi} \int_x^{\pi} \frac{\sin y}{y}\,dy\,dx$.

Solution

1. The function is integrated first w.r.t. y, but evaluation becomes easier by changing the order of integration.
2. Limits of y : $y = x$ to $y = \pi$
   Limits of x : $x = 0$ to $x = \pi$

3. The region is bounded by the line $y = x$, $y = \pi$ and $x = 0$.
4. The point of intersection of the line $y = x$ and the line $y = \pi$ is $P(\pi, \pi)$.
5. To change the order of integration, i.e., to integrate first w.r.t. x, draw a horizontal strip AB parallel to x-axis which starts from y-axis and terminates on the line $y = x$.

   Limits of $x : x = 0$ to $x = y$

   Limits of $y : y = 0$ to $y = \pi$

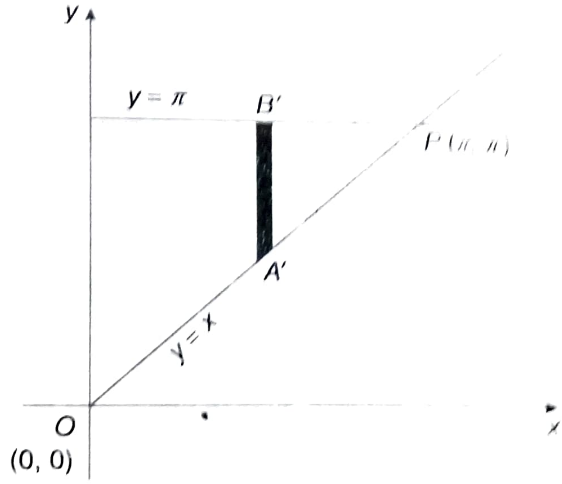

Fig. 5.33(a)

Hence, the given integral after change of order is

$$\int_0^{\pi}\int_x^{\pi} \frac{\sin y}{y}\,dy\,dx = \int_0^{\pi} \frac{\sin y}{y}\int_0^{y} dx\,dy$$

$$= \int_0^{\pi} \frac{\sin y}{y}\left|x\right|_0^{y} dy$$

$$= \int_0^{\pi} \frac{\sin y}{y}\cdot y\,dy$$

$$= \int_0^{\pi} \sin y\,dy$$

$$= \left|-\cos y\right|_0^{\pi}$$

$$= -\cos\pi + \cos 0$$

$$= 2$$

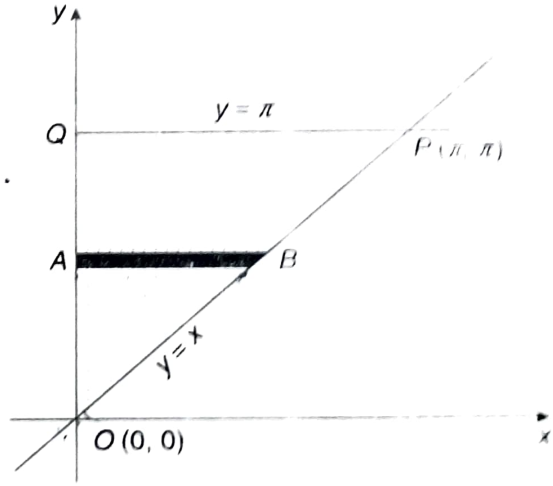

Fig. 5.33(b)

Example 3

Change the order of integration and evaluate $\int_0^{\infty}\int_x^{\infty} \frac{e^{-y}}{y}\,dy\,dx$.

Solution

1. Since inner limits depend on x, the function is integrated first w.r.t. y but evaluation becomes easier by changing the order of integration.
2. Limits of $y : y = x$ to $y = \infty$, along vertical strip

   Limits of $x : x = 0$ to $x = \infty$
3. The region is bounded by the lines $y = x$ and $x = 0$.
4. Here, the only point of intersection is origin.

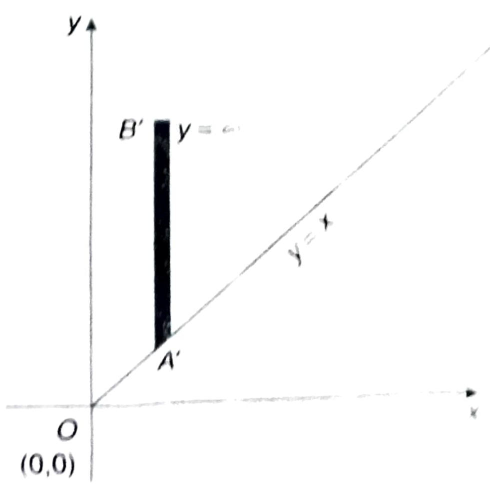

Fig. 5.34(a)

5. To change the order of integration, i.e. to integrate first w.r.t. x, draw a horizontal strip AB parallel to x-axis which starts from y-axis and terminates on the line $y = x$.
   Limits of $x : x = 0$ to $x = y$
   Limits of $y : y = 0$ to $y = \infty$

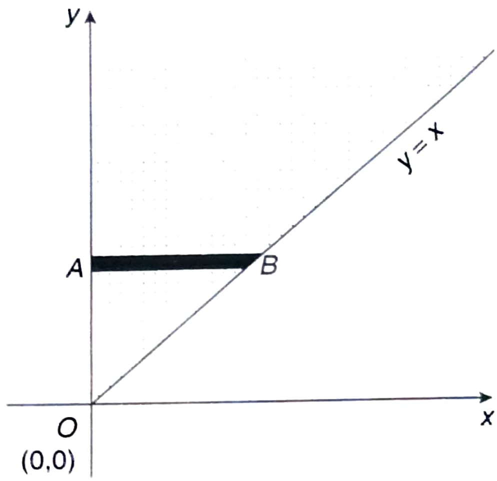

Fig. 5.34(b)

Hence, the given integral after change of order is

$$\int_0^\infty \int_x^\infty \frac{e^{-y}}{y}\, dy\, dx = \int_0^\infty \int_0^y \frac{e^{-y}}{y}\, dx\, dy$$

$$= \int_0^\infty \left\{\int_0^y dx\right\} \frac{e^{-y}}{y}\, dy$$

$$= \int_0^\infty |x|_0^y \frac{e^{-y}}{y}\, dy$$

$$= \int_0^\infty y \cdot \frac{e^{-y}}{y}\, dy$$

$$= \int_0^\infty e^{-y}\, dy$$

$$= \left|-e^{-y}\right|_0^\infty$$

$$= -(e^{-\infty} - e^0)$$

$$= 1$$

Example 4

Change the order of integration and evaluate $\int_0^\infty \int_0^x x e^{-\frac{x^2}{y}}\, dy\, dx.$

Solution

1. The function is integrated first w.r.t. y, but evaluation becomes easier by changing the order of integration.
2. Limits of $y : y = 0$ to $y = x$.
   Limits of $x : x = 0$ to $x \to \infty$.
3. The region is the part of the first quadrant bounded between the lines $y = x$ and $y = 0$.
4. To change the order of integration, i.e., to integrate first w.r.t. x, draw a horizontal strip parallel to x-axis which starts from the line $y = x$ and extends up to infinity.
   Limits of $x : x = y$ to $x \to \infty$
   Limits of $y : y = 0$ to $y \to \infty$

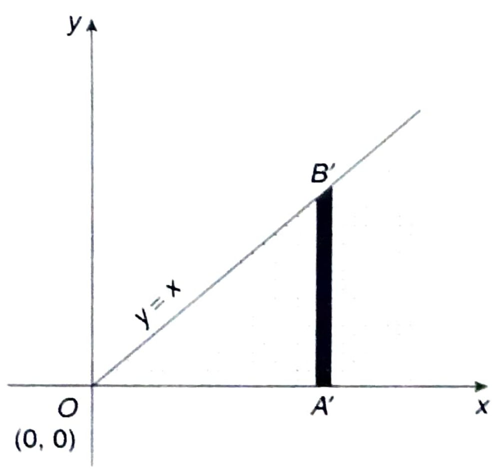

Fig. 5.35(a)

Hence, the given integral after change of order is

$$\int_0^\infty \int_0^x x e^{-\frac{x^2}{y}} \, dy \, dx = \int_0^\infty \int_y^\infty x e^{-\frac{x^2}{y}} \, dx \, dy$$

$$= \int_0^\infty \left(-\frac{y}{2}\right) \int_y^\infty e^{-\frac{x^2}{y}} \left(-\frac{2x}{y}\right) dx \, dy$$

$$= -\frac{1}{2} \int_0^\infty y \left| e^{-\frac{x^2}{y}} \right|_y^\infty dy$$

$$\left[\because \int e^{f(x)} f'(x) dx = e^{f(x)}\right]$$

$$= -\frac{1}{2} \int_0^\infty y(0 - e^{-y}) dy$$

$$= \frac{1}{2} \left| -ye^{-y} - e^{-y} \right|_0^\infty$$

$$= \frac{1}{2}$$

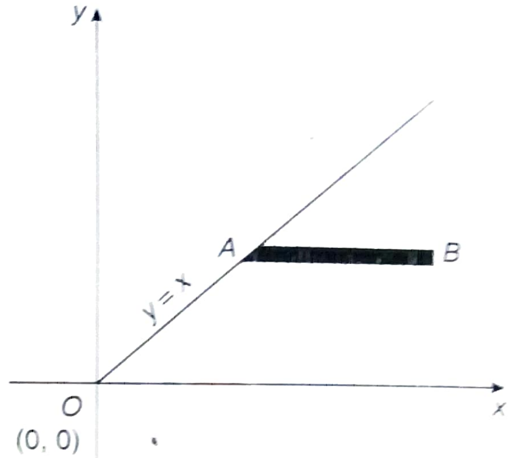

Fig. 5.35(b)

Example 5

Change the order of integration and evaluate

$$\int_0^a \int_y^a \frac{x^2}{\sqrt{x^2 + y^2}} \, dx \, dy.$$

Solution

1. Since inner limits depend on y, the function is integrated first w.r.t. x but evaluation becomes easier by changing the order of integration.
2. Limits of $x : x = y$ to $x = a$, along horizontal strip $A'B'$
   Limits of y : $y = 0$ to $y = a$
3. The region is bounded by the lines $y = x$, $x = a$ and $y = 0$.
4. The point of intersection of $y = x$ and $x = a$ is $x = a$, $y = a$.
   The point of intersection is $Q\ (a, a)$.
5. To change the order of integration, i.e. to integrate first w.r.t. y, draw a vertical strip AB parallel to y-axis which starts from x-axis and terminates on the line $y = x$.
   Limits of $y : y = 0$ to $y = x$
   Limits of x : $x = 0$ to $x = a$

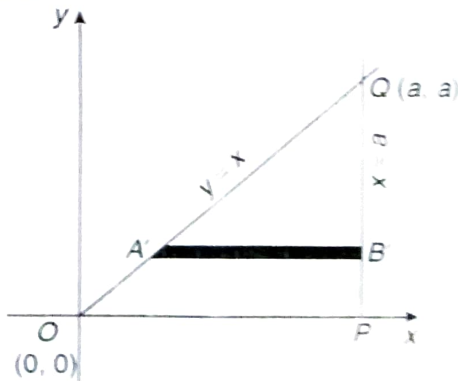

Fig. 5.36(a)

Hence, the given integral after change of order is

$$\int_0^a\int_y^a \frac{x^2}{\sqrt{x^2+y^2}}\,dx\,dy = \int_0^a\int_0^x \frac{x^2}{\sqrt{x^2+y^2}}\,dx$$

$$= \int_0^a\left\{\int_0^x \frac{1}{\sqrt{x^2+y^2}}\,dy\right\}x^2dx$$

$$= \int_0^a\left|\log\left(y+\sqrt{y^2+x^2}\right)\right|_0^x x^2dx$$

$$= \int_0^a\left[\log\left(x+\sqrt{2x^2}\right)-\log x\right]x^2dx$$

$$= \int_0^a\left[\log\frac{x\left(1+\sqrt{2}\right)}{x}\right]x^2dx$$

$$= \log\left(1+\sqrt{2}\right)\int_0^a x^2dx$$

$$= \log\left(1+\sqrt{2}\right)\left|\frac{x^3}{3}\right|_0^a$$

$$= \log\left(1+\sqrt{2}\right)\frac{a^3}{3}$$

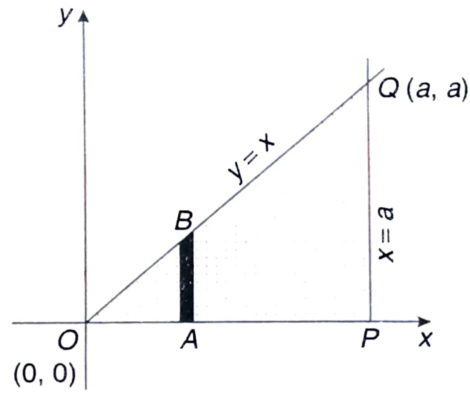

Fig. 5.36(b)

Example 6

Change the order of integration and evaluate

$$\int_0^1\int_0^{\sqrt{1-x^2}} \frac{e^y}{(e^y+1)\sqrt{1-x^2-y^2}}\,dy\,dx.$$

Solution

1. The function is integrated first w.r.t. y, but evaluation becomes easier by changing the order of integration.

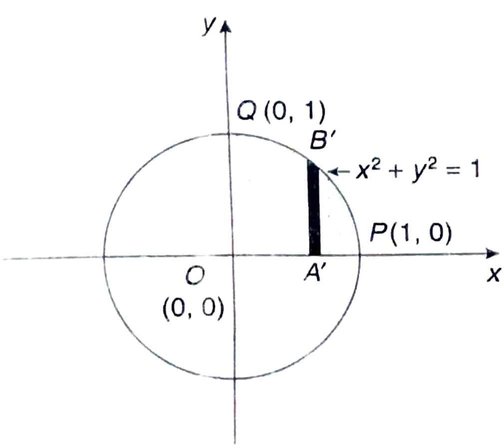

Fig. 5.37(a)

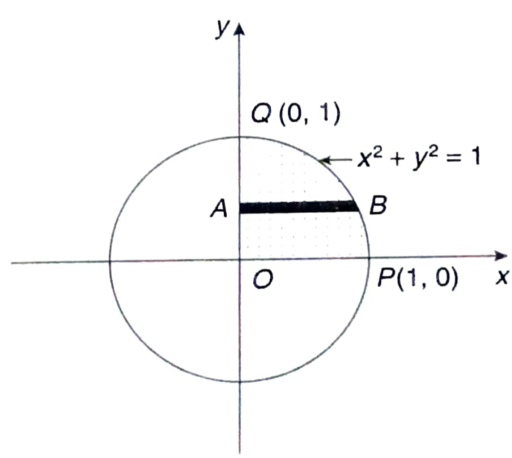

Fig. 5.37(b)

2. Limits of y : $y = 0$ to $y = \sqrt{1-x^2}$
   Limits of x : $x = 0$ to $x = 1$
3. Since given limits of x and y are positive, the region is the part of circle $x^2 + y^2 = 1$ in the first quadrant.
4. To change the order of integration, i.e., to integrate first w.r.t. x, draw a horizontal strip AB parallel to x-axis which starts from y axis and terminates on the circle $x^2 + y^2 = 1$.

   Limits of x : $x = 0$ to $x = \sqrt{1-y^2}$
   Limits of y : $y = 0$ to $y = 1$

Hence, the given integral after change of order is

$$\int_0^1\int_0^{\sqrt{1-x^2}} \frac{e^y}{(e^y+1)\sqrt{1-x^2-y^2}}\,dy\,dx = \int_0^1 \frac{e^y}{e^y+1}\int_0^{\sqrt{1-y^2}} \frac{1}{\sqrt{(1-y^2)-x^2}}\,dx\,dy$$

$$= \int_0^1 \frac{e^y}{e^y+1}\left|\sin^{-1}\frac{x}{\sqrt{1-y^2}}\right|_0^{\sqrt{1-y^2}} dy$$

$$= \int_0^1 \frac{e^y}{e^y+1}(\sin^{-1}1-\sin^{-1}0)dy$$

$$= \int_0^1 \frac{e^y}{e^y+1}\cdot\frac{\pi}{2}dy$$

$$= \frac{\pi}{2}\left|\log(e^y+1)\right|_0^1 \qquad \left[\because \int\frac{f'(y)}{f(y)}dy = \log f(y)\right]$$

$$= \frac{\pi}{2}[\log(e+1)-\log 2]$$

$$= \frac{\pi}{2}\log\left(\frac{e+1}{2}\right)$$

Example 7

Change the order of integration and evaluate $\int_0^a\int_{a-\sqrt{a^2-y^2}}^{a+\sqrt{a^2-y^2}} dx\,dy$.

Solution

1. Since inner limits depend on y, the function is integrated first w.r.t. x.
2. Limits of $x : x = a-\sqrt{a^2-y^2}$ to $x = a+\sqrt{a^2-y^2}$, along horizontal strip $A'B'$
   Limits of y : $y = 0$ to $y = a$
3. The region is bounded by the circle $(x-a)^2 + y^2 = a^2$ and the line $y = 0$. Since limits of y are positive, the region is the part of the circle $(x-a)^2 + y^2 = a^2$ above x-axis.
4. The points of intersection of the circle with x-axis are O (0, 0) and Q ($2a$, 0).
5. To change the order of integration i.e. to integrate first w.r.t. y, draw a vertical strip AB parallel to y-axis which starts from x-axis and terminates on the circle

$$(x-a)^2+y^2=a^2$$

or $$x^2+y^2-2ax=0$$

Limits of y : $y=0$ to $y=\sqrt{2ax-x^2}$

Limits of x : $x=0$ to $x=2a$

Hence, the given integral after change of order is

$$\int_0^a\int_{a-\sqrt{a^2-y^2}}^{a+\sqrt{a^2-y^2}} dx\,dy=\int_0^{2a}\int_0^{\sqrt{2ax-x^2}} dy\,dx$$

$$=\int_0^{2a}|y|_0^{\sqrt{2ax-x^2}}\,dx$$

$$=\int_0^{2a}\sqrt{2ax-x^2}\,dx$$

$$=\int_0^{2a}\sqrt{a^2-(x-a)^2}$$

$$=\left|\frac{(x-a)}{2}\sqrt{a^2-(x-a)^2}+\frac{a^2}{2}\sin^{-1}\left(\frac{x-a}{a}\right)\right|_0^{2a}$$

$$=\frac{a}{2}\sqrt{0}+\frac{a^2}{2}\sin^{-1}1-\frac{(0-a)}{2}\sqrt{0}-\frac{a^2}{2}\sin^{-1}(-1)$$

$$=a^2\sin^{-1}1 \quad [\because \sin^{-1}(-1)=-\sin^{-1}(1)]$$

$$=a^2\frac{\pi}{2}$$

$$=\frac{\pi a^2}{2}$$

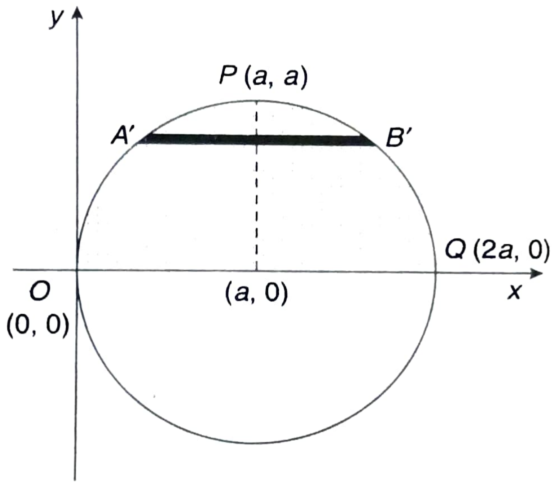

Fig. 5.38(a)

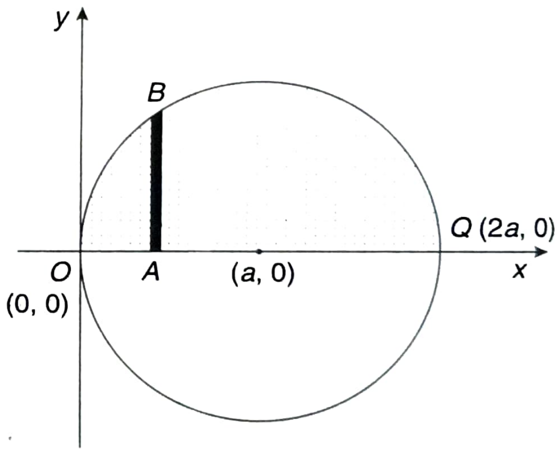

Fig. 5.38(b)

Example 8

Change the order of integration and evaluate $\int_0^1\int_{x^2}^{2-x} xy\,dx\,dy$.

Solution

1. Since inner limits depend on x, the function is integrated first w.r.t. y.
   The correct form of integral
   $$=\int_0^1\int_{x^2}^{2-x} xy\,dy\,dx$$
2. Limits of y : $y=x^2$ to $y=2-x$, along vertical strip $A'B'$
   Limits of x : $x=0$ to $x=1$
3. The region is bounded by y-axis, the line $x+y=2$ and the parabola $x^2=y$. Since given limits of x and y are positive, the region lies in the first quadrant.

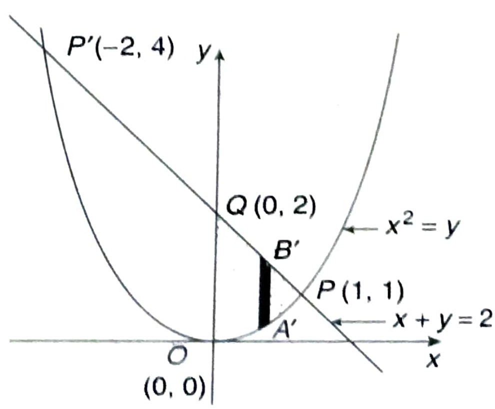

Fig. 5.39(a)

4. The points of intersection of $x + y = 2$ and $x^2 = y$ are obtained as

$$x^2 = 2 - x$$
$$x^2 + x - 2 = 0$$
$$(x - 1)(x + 2) = 0$$
$$x = 1, -2$$
$$y = 1, 4$$

The points of intersection are $P(1, 1)$ and $P'(-2, 4)$.

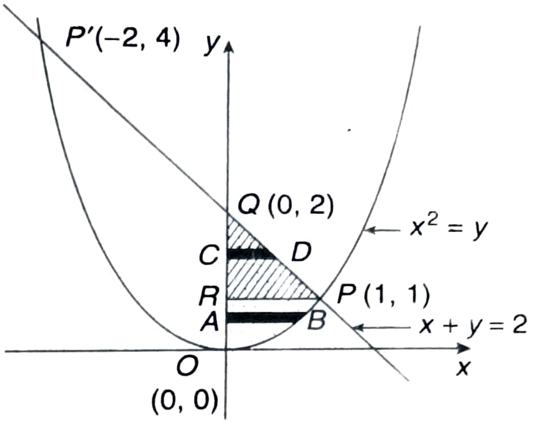

Fig. 5.39(b)

5. To change the order of integration, i.e., to integrate first w.r.t. x, divide the region into two subregions OPR and RPQ. Draw a horizontal strip parallel to x-axis in each subregion.
   (i) In subregion OPR, strip AB starts from y-axis and terminates on the parabola $x^2 = y$.
   Limits of x : $x = 0$ to $x = \sqrt{y}$
   Limits of y : $y = 0$ to $y = 1$
   (ii) In subregion RPQ, strip CD starts from y-axis and terminates on the line $x + y = 2$.
   Limits of x : $x = 0$ to $x = 2 - y$
   Limits of y : $y = 1$ to $y = 2$

Hence, the given integral after change of order is

$$\int_0^1 \int_{x^2}^{2-x} xy \, dy \, dx = \int_0^1 \int_0^{\sqrt{y}} xy \, dx \, dy + \int_1^2 \int_0^{2-y} xy \, dx \, dy$$
$$= \int_0^1 \left| \frac{x^2}{2} \right|_0^{\sqrt{y}} y \, dy + \int_1^2 \left| \frac{x^2}{2} \right|_0^{2-y} y \, dy$$
$$= \frac{1}{2} \int_0^1 (y) y \, dy + \frac{1}{2} \int_1^2 (2 - y)^2 y \, dy$$
$$= \frac{1}{2} \int_0^1 y^2 dy + \frac{1}{2} \int_1^2 (4y - 4y^2 + y^3) dy$$
$$= \frac{1}{2} \left| \frac{y^3}{3} \right|_0^1 + \frac{1}{2} \left| 4\frac{y^2}{2} - 4\frac{y^3}{3} + \frac{y^4}{4} \right|_1^2$$
$$= \frac{1}{2}\left(\frac{1}{3}\right) + \frac{1}{2}\left(8 - \frac{32}{3} + 4 - 2 + \frac{4}{3} - \frac{1}{4}\right)$$
$$= \frac{1}{6} + \frac{5}{24}$$
$$= \frac{9}{24}$$
$$= \frac{3}{8}$$

Example 9

Change the order of integration and evaluate $\int_0^{4a}\int_{\frac{x^2}{4a}}^{2\sqrt{ax}} xy\,dy\,dx.$

Solution

1. Since inner limits depend on x, the function is integrated first w.r.t. y.
2. Limits of $y: y=\frac{x^2}{4a}$ to $y=2\sqrt{ax}$, along vertical strip $A'B'$
3. The region is bounded by the parabolas $x^2 = 4ay$ and $y^2 = 4ax$.
4. The points of intersection of $x^2 = 4ay$ and $y^2 = 4ax$ are obtained as

$$x^4 = 16a^2y^2$$
$$=16a^2(4ax)$$
$$x(x^3 - 64a^3) = 0$$
$$x = 0, x = 4a$$
$$\therefore y = 0, y = 4a$$

The points of intersection are O (0, 0) and P (4a, 4a).

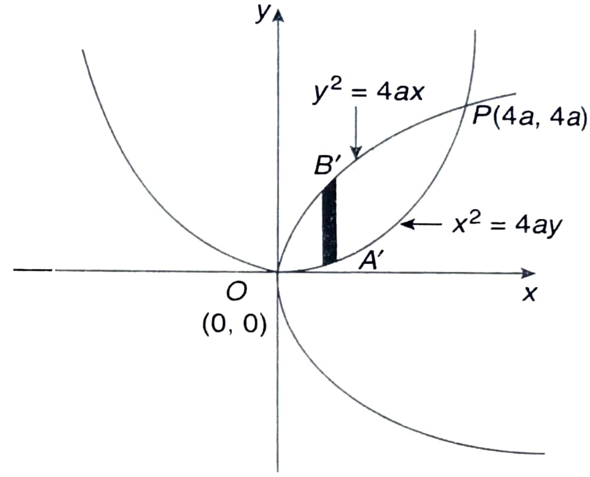

Fig. 5.40(a)

5. To change the order of integration, i.e., to integrate first w.r.t. x, draw a horizontal strip AB parallel to x-axis which starts from the parabola $y^2 = 4ax$ and terminates on the parabola $x^2 = 4ay$.

Limits of $x: x=\frac{y^2}{4a}$ to $x=2\sqrt{ay}$

Limits of $y: y=0$ to $y=4a$

Hence, the given integral after change of order is

$$\int_0^{4a}\int_{\frac{x^2}{4a}}^{2\sqrt{ax}} xy\,dy\,dx = \int_0^{4a}\int_{\frac{y^2}{4a}}^{2\sqrt{ay}} xy\,dx\,dy$$

$$= \int_0^{4a}\left|\frac{x^2}{2}\right|_{\frac{y^2}{4a}}^{2\sqrt{ay}} y\,dy$$

$$= \frac{1}{2}\int_0^{4a}\left(4ay - \frac{y^4}{16a^2}\right)dy$$

$$= \frac{1}{2}\left|4a\cdot\frac{y^2}{2} - \frac{1}{16a^2}\cdot\frac{y^5}{5}\right|_0^{4a}$$

$$= \frac{1}{2}\left[2a(16a^2) - \frac{1}{16a^2}\cdot\frac{(4a)^5}{5}\right]$$

$$= \frac{1}{2}\cdot\frac{96}{5}a^3$$

$$= \frac{48}{5}a^3$$

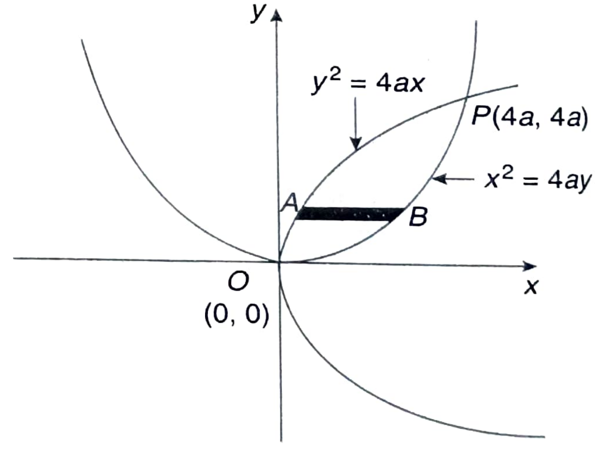

Fig. 5.40(b)

Example 10

Change the order of integration and evaluate

$$\int_0^a \int_0^{a-\sqrt{a^2-y^2}} \frac{xy\log(x+a)}{(x-a)^2}\,dx\,dy.$$

Solution

1. The function is integrated first w.r.t. x, but evaluation becomes easier by changing the order of integration.
2. Limits of x : $x = 0$ to $x = a - \sqrt{a^2 - y^2}$

   Limits of y : $y = 0$ to $y = a$
3. The region is bounded by the circle $(x-a)^2 + y^2 = a^2$, the lines $y = a$ and $x = 0$.
4. The point of intersection of $(x-a)^2 + y^2 = a^2$ and $y = a$ is obtained as

   $$(x-a)^2 + a^2 = a^2$$
   $$x = a$$

   The point of intersection is $P(a, a)$.

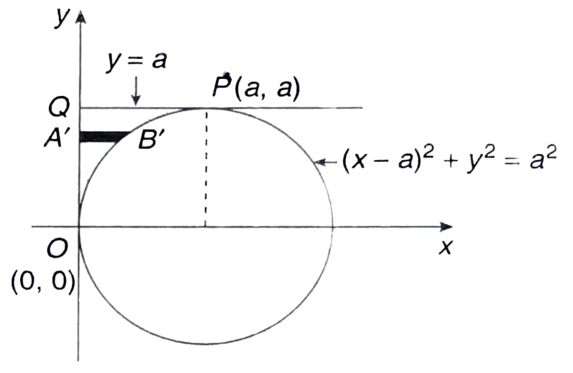

Fig. 5.41(a)

5. To change the order of integration, i.e., to integrate first w.r.t. y, draw a vertical strip AB parallel to y-axis which starts from the circle $(x-a)^2 + y^2 = a^2$ and terminates on the line $y = a$.

   Limits of y: $y = \sqrt{2ax - x^2}$ to $y = a$

   Limits of x : $x = 0$ to $x = a$

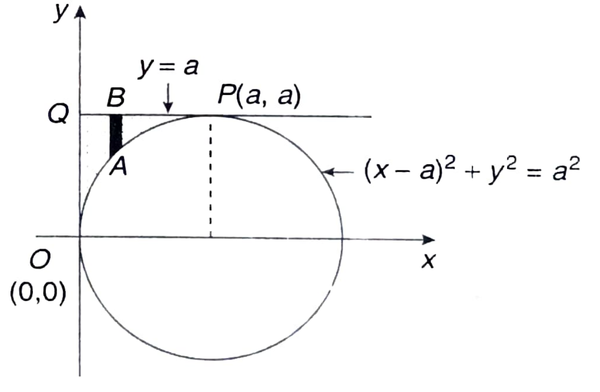

Fig. 5.41(b)

Hence, the given integral after change of order is

$$\int_0^a \int_0^{a-\sqrt{a^2-y^2}} \frac{xy\log(x+a)}{(x-a)^2}\,dx\,dy = \int_0^a \int_{\sqrt{2ax-x^2}}^a \frac{x\log(x+a)}{(x-a)^2}\,y\,dy\,dx$$

$$= \int_0^a \frac{x\log(x+a)}{(x-a)^2} \left| \frac{y^2}{2} \right|_{\sqrt{2ax-x^2}}^a dx$$

$$= \int_0^a \frac{x\log(x+a)}{(x-a)^2} \left(\frac{a^2 - 2ax + x^2}{2} \right) dx$$

$$= \frac{1}{2} \int_0^a x\log(x+a)\,dx$$

$$= \frac{1}{2} \left[\left| \frac{x^2}{2} \log(x+a) \right|_0^a - \int_0^a \frac{x^2}{2} \cdot \frac{1}{x+a}\,dx \right]$$

$$= \frac{1}{2}\left[\frac{a^2}{2}\log 2a - \frac{1}{2}\int_0^a \left\{(x-a) + \frac{a^2}{x+a}\right\} dx\right]$$

$$= \frac{1}{2}\left[\frac{a^2}{2}\log 2a - \frac{1}{2}\left|\frac{x^2}{2} - ax + a^2\log(x+a)\right|_0^a\right]$$

$$= \frac{1}{4}\left(a^2\log 2a - \frac{a^2}{2} + a^2 - a^2\log 2a + a^2\log a\right)$$

$$= \frac{1}{4}\left(\frac{a^2}{2} + a^2\log a\right)$$

$$= \frac{a^2}{8}(1 + 2\log a)$$

Example 11

Change the order of integration and evaluate

$$\int_0^{\frac{1}{2}}\int_0^{\sqrt{1-4y^2}} \frac{1+x^2}{\sqrt{1-x^2}\sqrt{1-x^2-y^2}} dx\, dy.$$

Solution

1. The function is integrated first w.r.t. x, but evaluation becomes easier by changing the order of integration.
2. Limits of x : $x = 0$ to $x = \sqrt{1-4y^2}$

   Limits of y : $y = 0$ to $y = \frac{1}{2}$
3. Since the limits of x and y are positive, the region is the part of the ellipse in the first quadrant.

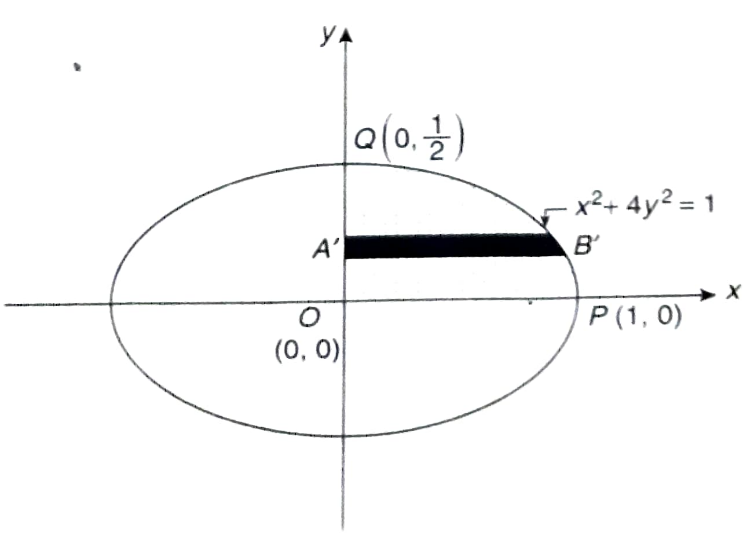

Fig. 5.42(a)

4. To change the order of integration, i.e., to integrate first w.r.t. y, draw a vertical strip AB parallel to y-axis which starts from x-axis and terminates on the ellipse $x^2 + 4y^2 = 1$.

Limits of y : $y = 0$ to $y = \frac{1}{2}\sqrt{1-x^2}$

Limits of x : $x = 0$ to $x = 1$

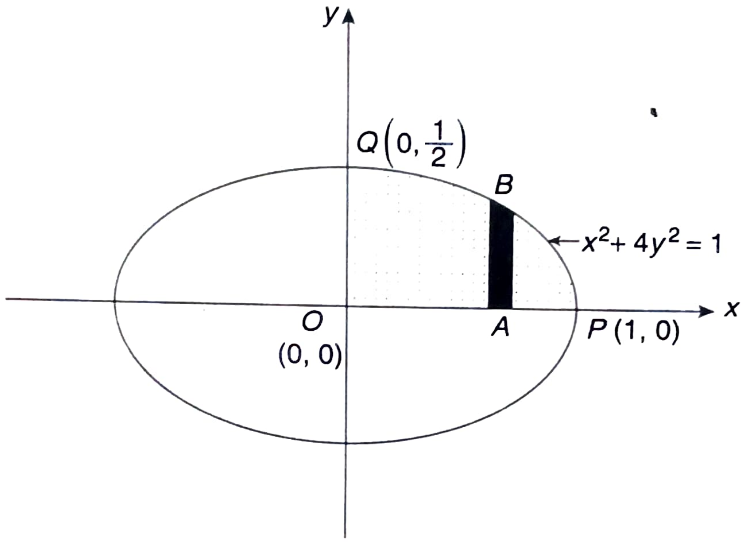

Fig. 5.42(b)

Hence, the given integral after change of order is

$$\int_0^{\frac{1}{2}}\int_0^{\sqrt{1-4y^2}} \frac{1+x^2}{\sqrt{1-x^2}\sqrt{1-x^2-y^2}}\,dx\,dy = \int_0^1 \frac{1+x^2}{\sqrt{1-x^2}}\int_0^{\frac{1}{2}\sqrt{1-x^2}} \frac{1}{\sqrt{(1-x^2)-y^2}}\,dy\,dx$$

$$= \int_0^1 \frac{1+x^2}{\sqrt{1-x^2}}\left|\sin^{-1}\frac{y}{\sqrt{1-x^2}}\right|_0^{\frac{1}{2}\sqrt{1-x^2}} dx$$

$$= \int_0^1 \frac{1+x^2}{\sqrt{1-x^2}}\left(\sin^{-1}\frac{1}{2} - \sin^{-1}0\right)dx$$

$$= \int_0^1 \frac{2-(1-x^2)}{\sqrt{1-x^2}}\cdot\frac{\pi}{6}\,dx$$

$$= \frac{\pi}{6}\int_0^1\left(\frac{2}{\sqrt{1-x^2}} - \sqrt{1-x^2}\right)dx$$

$$= \frac{\pi}{6}\left|2\sin^{-1}x - \frac{x}{2}\sqrt{1-x^2} - \frac{1}{2}\sin^{-1}x\right|_0^1$$

$$\left[\because \int\sqrt{a^2-x^2}\,dx = \frac{x}{2}\sqrt{a^2-x^2} + \frac{a^2}{2}\sin^{-1}\frac{x}{a}\right]$$

$$= \frac{\pi}{6}\left(\frac{3}{2}\sin^{-1} 1\right)$$

$$= \frac{\pi}{4} \cdot \frac{\pi}{2}$$

$$= \frac{\pi^2}{8}$$

Example 12

Change the order of integration and evaluate

$$\int_0^a \int_0^y \frac{x\,dy\,dx}{\sqrt{(a^2 - x^2)(a-y)(y-x)}}.$$

Solution

1. The function is integrated first w.r.t. x, but evaluation becomes easier by changing the order of integration.
2. Limits of $x : x = 0$ to $x = y$
   Limits of $y : y = 0$ to $y = a$
3. The region is bounded by the line $y = x$, $y = a$ and $x = 0$.
4. The point of intersection of $y = a$ and $y = x$ is $P\ (a, a)$.
5. To change the order of integration, i.e., to integrate first w.r.t. y, draw a vertical strip AB parallel to y-axis which starts from the line $y = x$ and terminates on the line $y = a$.
   Limits of $y : y = x$ to $y = a$
   Limits of $x : x = 0$ to $x = a$

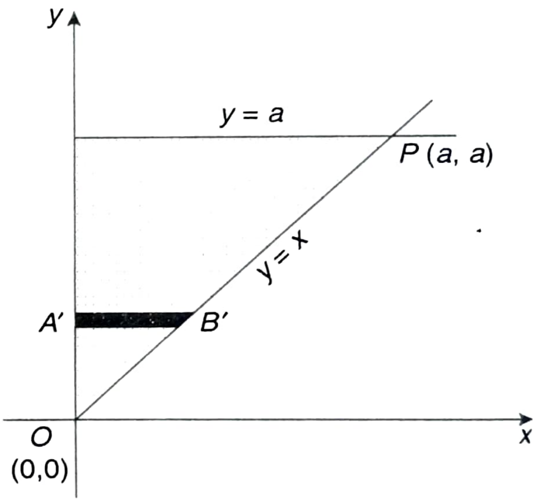

Fig. 5.43(a)

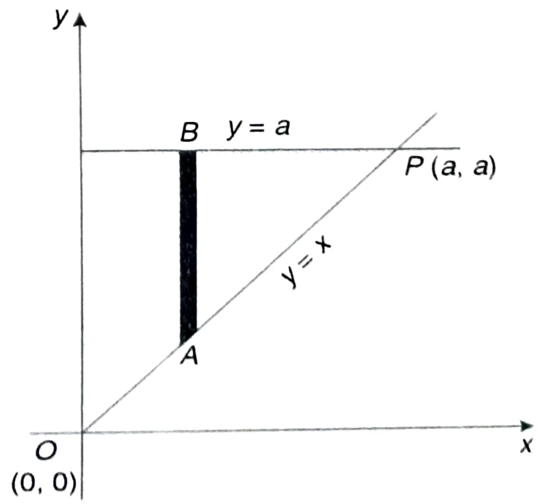

Fig. 5.43(b)

Hence, the given integral after change of order is

$$\int_0^a \int_0^y \frac{x\,dy\,dx}{\sqrt{(a^2 - x^2)(a-y)(y-x)}}$$

$$= \int_0^a \int_x^a \frac{x\,dy\,dx}{\sqrt{(a^2 - x^2)(a-y)(y-x)}}$$

$$= \int_0^a \frac{x}{\sqrt{a^2 - x^2}} \left[\int_x^a \frac{dy}{\sqrt{(a-y)(y-x)}}\right] dx$$

Putting $y - x = t^2$, $dy = 2t\,dt$

When $\quad y = x, \quad t = 0$

When $\quad y = a, \quad t = \sqrt{a - x}$

$$\int_0^a \int_0^y \frac{x\,dy\,dx}{\sqrt{(a^2-x^2)(a-y)(y-x)}} = \int_0^a \frac{x}{\sqrt{a^2-x^2}} \int_0^{\sqrt{a-x}} \frac{2t\,dt}{\sqrt{(a-x-t^2)t^2}}\,dx$$

$$= 2\int_0^a \frac{x}{\sqrt{a^2-x^2}} \int_0^{\sqrt{a-x}} \frac{dt}{\sqrt{(a-x)-t^2}}\,dx$$

$$= 2\int_0^a \frac{x}{\sqrt{a^2-x^2}} \left| \sin^{-1}\frac{t}{\sqrt{a-x}} \right|_0^{\sqrt{a-x}} dx$$

$$= 2\int_0^a \frac{x}{\sqrt{a^2-x^2}} (\sin^{-1}1 - \sin^{-1}0)\,dx$$

$$= 2\cdot\frac{\pi}{2}\int_0^a \left[-\frac{1}{2}(a^2-x^2)^{-\frac{1}{2}}(-2x) \right] dx$$

$$= -\frac{\pi}{2}\left| 2(a^2-x^2)^{\frac{1}{2}} \right|_0^a \quad \left[\because \int [f(x)]^n f'(x)\,dx = \frac{[f(x)]^{n+1}}{n+1} \right]$$

$$= \pi a$$

Example 13

Change the order of integration and evaluate

$$\int_0^1 \int_x^{\frac{1}{x}} \frac{y}{(1+xy)^2(1+y^2)}\,dy\,dx.$$

Solution

1. The function is integrated first w.r.t. y, but evaluation becomes easier by changing the order of integration.
2. Limits of y : $y = x$ to $y = \frac{1}{x}$

   Limits of x : $x = 0$ to $x = 1$
3. The region is bounded by the rectangular hyperbola $xy = 1$, the line $y = x$ and y-axis in the first quadrant.
4. The point of intersection of $xy = 1$ and $y = x$ in the first quadrant is obtained as

$$x^2 = 1$$
$$x = 1$$
$$\therefore y = 1$$

   The point of intersection is $P(1, 1)$.
5. To change the order of integration, i.e., to integrate first w.r.t. x, divide the region into two subregions OPQ and QPR. Draw a horizontal strip parallel to x-axis in each subregion.

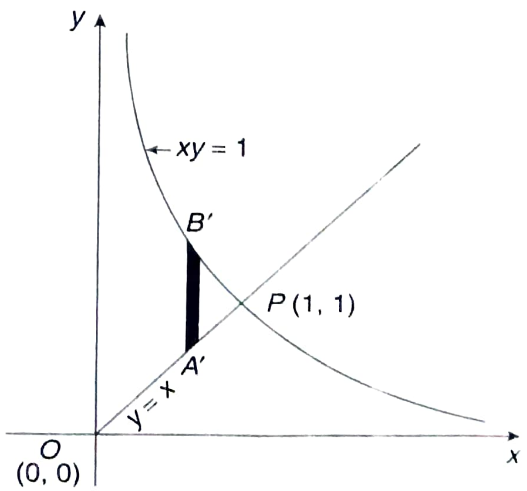

Fig. 5.44(a)

(i) In subregion OPQ, strip AB starts from y-axis and terminates on the line $y = x$.

Limits of $x : x = 0$ to $x = y$
Limits of $y : y = 0$ to $y = 1$

(ii) In subregion QPR, strip CD starts from y-axis and terminates on the rectangular hyperbola $xy = 1$.

Limits of $x : x = 0$ to $x = \dfrac{1}{y}$

Limits of $y : y = 1$ to $y \to \infty$.

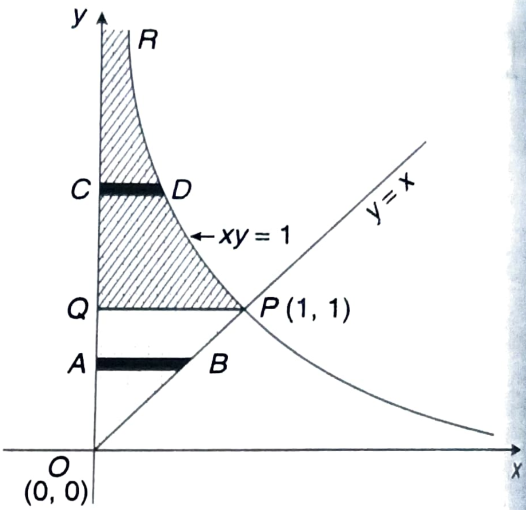

Fig. 5.44(b)

Hence, the given integral after change of order is

$$\int_0^1 \int_x^{\frac{1}{x}} \frac{y}{(1+xy)^2(1+y^2)}\,dy\,dx = \int_0^1 \int_0^y \frac{y}{1+y^2}\frac{1}{(1+xy)^2}\,dx\,dy + \int_1^\infty \int_0^{\frac{1}{y}} \frac{y}{1+y^2}\frac{1}{(1+xy)^2}\,dx\,dy$$

$$= \int_0^1 \frac{y}{1+y^2}\left|-\frac{1}{y(1+xy)}\right|_0^y dy + \int_1^\infty \frac{y}{1+y^2}\left|-\frac{1}{y(1+xy)}\right|_0^{\frac{1}{y}} dy$$

$$= -\int_0^1 \frac{1}{1+y^2}\left(\frac{1}{1+y^2} - 1\right)dy - \int_1^\infty \frac{1}{1+y^2}\left(\frac{1}{2} - 1\right)dy$$

$$= -\int_0^1 \left[\frac{1}{(1+y^2)^2} - \frac{1}{1+y^2}\right]dy + \frac{1}{2}\int_1^\infty \frac{1}{1+y^2}\,dy$$

Putting $y = \tan\theta$ in the first term of first integral, $dy = \sec^2\theta\, d\theta$,
When $y = 0$, $\theta = 0$
When $y = 1$, $\theta = \dfrac{\pi}{4}$

$$\int_0^1 \int_x^{\frac{1}{x}} \frac{y}{(1+xy)^2(1+y^2)}\,dy\,dx = -\int_0^{\frac{\pi}{4}} \frac{\sec^2\theta\, d\theta}{\sec^4\theta} + \left|\tan^{-1} y\right|_0^1 + \frac{1}{2}\left|\tan^{-1} y\right|_1^\infty$$

$$= -\int_0^{\frac{\pi}{4}} \frac{(1+\cos 2\theta)}{2}\,d\theta + (\tan^{-1} 1 - \tan^{-1} 0) + \frac{1}{2}(\tan^{-1}\infty - \tan^{-1} 1)$$

$$= -\frac{1}{2}\left|\theta + \frac{\sin 2\theta}{2}\right|_0^{\frac{\pi}{4}} + \frac{\pi}{4} + \frac{1}{2}\left(\frac{\pi}{2} - \frac{\pi}{4}\right)$$

$$= -\frac{\pi}{8} - \frac{1}{4}\sin\frac{\pi}{2} + \frac{3\pi}{8}$$

$$= \frac{\pi - 1}{4}$$

Example 14

Change the order of integration and evaluate

$$\int_0^1 \int_0^{\sqrt{1-y^2}} \frac{\cos^{-1} x}{\sqrt{1-x^2}\sqrt{1-x^2-y^2}}\, dx\, dy.$$

Solution

1. The function is integrated first w.r.t. x, but evaluation becomes easier by changing the order of integration.
2. Limits of $x : x = 0$ to $x = \sqrt{1-y^2}$
   Limits of $y : y = 0$ to $y = 1$
3. Since given limits of x and y are positive, the region is the part of the circle $x^2 + y^2 = 1$ in the first quadrant.

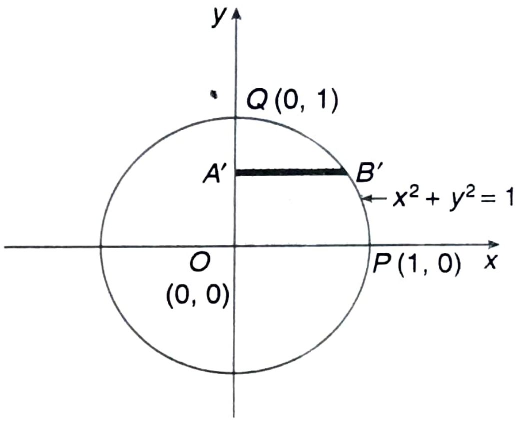

Fig. 5.45(a)

4. To change the order of integration, i.e., to integrate first w.r.t. y, draw a vertical strip AB parallel to y-axis in the region. AB starts from x-axis and terminates on the circle $x^2 + y^2 = 1$.
   Limits of $y : y = 0$ to $y = \sqrt{1-x^2}$
   Limits of $x : x = 0$ to $x = 1$.

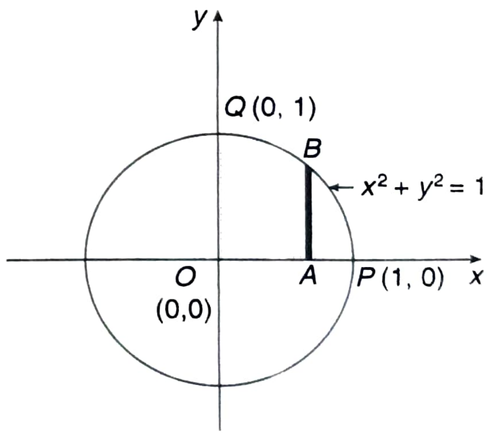

Fig. 5.45(b)

Hence, the given integral after change of order is

$$\int_0^1 \int_0^{\sqrt{1-y^2}} \frac{\cos^{-1} x}{\sqrt{1-x^2}\sqrt{1-x^2-y^2}}\, dx\, dy$$

$$= \int_0^1 \int_0^{\sqrt{1-x^2}} \frac{\cos^{-1} x}{\sqrt{1-x^2}} \frac{1}{\sqrt{(1-x^2)-y^2}}\, dy\, dx$$

$$= \int_0^1 \frac{\cos^{-1} x}{\sqrt{1-x^2}} \left| \sin^{-1} \frac{y}{\sqrt{1-x^2}} \right|_0^{\sqrt{1-x^2}} dx$$

$$= \int_0^1 \frac{\cos^{-1} x}{\sqrt{1-x^2}} (\sin^{-1} 1 - \sin^{-1} 0) dx$$

$$= -\frac{\pi}{2} \int_0^1 \cos^{-1} x \left(-\frac{1}{\sqrt{1-x^2}} \right) dx$$

$$= -\frac{\pi}{2} \left| \frac{(\cos^{-1} x)^2}{2} \right|_0^1 \qquad \left[\because \int f(x) f'(x) dx = \frac{[f(x)]^2}{2} \right]$$

$$= -\frac{\pi}{4} \left[(\cos^{-1} 1)^2 - (\cos^{-1} 0)^2 \right]$$

$$= -\frac{\pi}{4} \left[0 - \left(\frac{\pi}{2} \right)^2 \right]$$

$$= \frac{\pi^3}{16}$$

EXERCISE 5.3

(I) Change the order of integration of the following:

1. $\int_0^6 \int_{2-x}^{2+x} f(x, y)dy\,dx$

$$\left[\textbf{Ans.:} \int_{-4}^{2} \int_{2-y}^{6} f(x, y)dy\,dx + \int_{2}^{8} \int_{y-2}^{6} f(x, y)dy\,dx\right]$$

2. $\int_0^1 \int_x^{2x} f(x, y)dy\,dx$

$$\left[\textbf{Ans.:} \int_0^1 \int_{\frac{y}{2}}^{y} f(x, y)dx\,dy + \int_1^2 \int_{\frac{y}{2}}^{1} f(x, y)dx\,dy\right]$$

3. $\int_0^1 \int_{-\sqrt{y}}^{\sqrt{y}} f(x, y)dx\,dy$

$$\left[\textbf{Ans.:} \int_{-1}^{1} \int_{x^2}^{1} f(x, y)dx\,dy\right]$$

4. $\int_{-a}^{a} \int_0^{\frac{y^2}{a}} f(x, y)dx\,dy$

$$\left[\textbf{Ans.:} \int_0^a \int_{-a}^{-\sqrt{ax}} f(x, y)dy\,dx + \int_0^a \int_{\sqrt{ax}}^{a} f(x, y)dy\,dx\right]$$

5. $\int_0^a \int_{\frac{x^2}{a}}^{2a-x} f(x, y)dy\,dx$

$$\left[\textbf{Ans.:} \int_0^a \int_0^{\sqrt{ay}} f(x, y)dx\,dy + \int_a^{2a} \int_0^{2a-y} f(x, y)dx\,dy\right]$$

6. $\int_{-2}^{3} \int_{y^2-6}^{y} f(x, y)dx\,dy$

$$\left[\textbf{Ans.:} \int_{-6}^{-2} \int_{-\sqrt{x+6}}^{\sqrt{x+6}} f(x, y)dy\,dx + \int_{-2}^{3} \int_{x}^{\sqrt{x+6}} f(x, y)dy\,dx\right]$$

7. $\int_0^1 \int_{2y}^{2(1-\sqrt{1-y})} f(x, y)\,dx\,dy$

$$\left[\textbf{Ans.:} \int_0^2 \int_0^{\frac{x}{2}} f(x, y)dy\,dx + \int_2^4 \int_0^{\frac{4x-x^2}{4}} f(x, y)dy\,dx\right]$$

8. $\int_0^2 \int_{\frac{x^2}{4}-4}^{\frac{6-x}{2}} f(x, y)\,dy\,dx$

$$\left[\textbf{Ans.:} \int_1^2 \int_0^{2\sqrt{y-1}} f(x, y)dx\,dy + \int_2^3 \int_0^{6-2y} f(x, y)dx\,dy\right]$$

9. $\int_0^2 \int_{\sqrt{4-x^2}}^{x+6a} f(x, y)dy\,dx$

$$\left[\text{Ans.: } \int_0^2 \int_{\sqrt{4-y^2}}^{2} f(x, y)dx\,dy + \int_2^{6a} \int_0^2 f(x, y)dx\,dy + \int_{6a}^{6a+2} \int_{y-6a}^{2} f(x, y)dx\,dy\right]$$

10. $\int_0^a \int_{\sqrt{a^2-y^2}}^{y+a} f(x, y)dx\,dy$

$$\left[\text{Ans.: } \int_0^a \int_{\sqrt{a^2-x^2}}^{a} f(x, y)dy\,dx + \int_a^{2a} \int_{x-a}^{a} f(x, y)dy\,dx\right]$$

11. $\int_0^{2a} \int_{\sqrt{2ax-x^2}}^{\sqrt{2ax}} f(x, y)dx\,dy$

$$\left[\text{Ans.: } \int_0^a \int_{a+\sqrt{a^2-y^2}}^{2a} f(x, y)dx\,dy + \int_0^a \int_{\frac{y^2}{2a}}^{a-\sqrt{a^2-y^2}} f(x, y)dx\,dy + \int_a^{2a} \int_{\frac{y^2}{2a}}^{2a} f(x, y)dx\,dy\right]$$

12. $\int_0^a \int_{\sqrt{\frac{a^2-x^2}{4}}}^{\sqrt{a^2-x^2}} f(x, y)dy\,dx$

$$\left[\text{Ans.: } \int_0^{\frac{a}{2}} \int_{\sqrt{a^2-4y^2}}^{\sqrt{a^2-y^2}} f(x, y)dx\,dy + \int_{\frac{a}{2}}^{a} \int_0^{\sqrt{a^2-y^2}} f(x, y)dx\,dy\right]$$

13. $\int_0^a \int_x^{\frac{a^2}{x}} f(x, y)dy\,dx$

$$\left[\text{Ans.: } \int_0^a \int_0^y f(x, y)dx\,dy + \int_a^{\infty} \int_0^{\frac{a^2}{y}} f(x, y)dx\,dy\right]$$

14. $\int_a^b \int_{\frac{k}{x}}^{mx} f(x, y)dy\,dx$

$$\left[\text{Ans.: } \int_{\frac{k}{b}}^{\frac{k}{a}} \int_{\frac{k}{y}}^{b} f(x, y)dx\,dy + \int_{\frac{k}{a}}^{ma} \int_a^b f(x, y)dx\,dy + \int_{ma}^{mb} \int_{\frac{y}{m}}^{b} f(x, y)\,dx\,dy\right]$$

15. $\int_0^1 \int_1^{e^x} f(x, y)dy\,dx$

$$\left[\text{Ans.: } \int_1^e \int_{\log y}^{1} f(x, y)dx\,dy\right]$$

16. $\int_0^2 \int_0^{x^3} f(x, y)dy\,dx$

$$\left[\text{Ans.: } \int_0^8 \int_{y^{\frac{1}{3}}}^{2} f(x, y)dx\,dy\right]$$

17. $\int_0^{\frac{1}{2}}\int_0^{\sqrt{1-4y^2}} \frac{1+x^2}{\sqrt{1-x^2-y^2}}\,dx\,dy$

$$\left[\text{Ans.: } \int_0^1\int_0^{\frac{\sqrt{1-x^2}}{2}} \frac{1+x^2}{\sqrt{1-x^2-y^2}}\,dy\,dx = \frac{2\pi}{3}\right]$$

18. $\int_0^2\int_0^{\frac{x^2}{2}} \frac{x}{\sqrt{x^2+y^2+1}}\,dy\,dx$

$$\left[\text{Ans.: } \int_0^2\int_{\sqrt{2y}}^2 \frac{x}{\sqrt{x^2+y^2+1}}\,dx\,dy = \frac{1}{4}(5\log 5-4)\right]$$

19. $\int_0^a\int_y^a \frac{x}{x^2+y^2}\,dy\,dx$

$$\left[\text{Ans.: } \frac{\pi a}{4}\right]$$

5.4 DOUBLE INTEGRALS IN POLAR COORDINATES

The integral $\int_{\theta_1}^{\theta_2}\int_{r_1}^{r_2} f(r,\theta)\,dr\,d\theta$ represents the polar form of the double integration. This integral is first integrated w.r.t. r keeping θ constant and then the resulting expression is integrated w.r.t. θ.

5.4.1 Limits of Integration

If the limits of integration are not given then these limits are obtained from the equations of the given curves. Let the region is bounded by the curves $r = r_1(\theta)$, $r = r_2(\theta)$ and the lines $\theta = \theta_1$, $\theta = \theta_2$.

The region of integration is *PQRS*. Draw an elementary radius vector *AB* from origin which enters in the region from the curve $r = r_1(\theta)$ and leaves at the curve $r = r_2(\theta)$. Therefore, limits for r are $r_1(\theta)$ to $r_2(\theta)$ [i.e., r varies along the *AB* and θ remains constant]

To cover the entire region *PQRS*, rotate elementary radius vector *AB* from *PQ* to *RS*. Therefore, θ varies from θ_1 to θ_2.

$$\iint f(r,\theta)\,dr\,d\theta = \int_{\theta_1}^{\theta_2}\int_{r_1(\theta)}^{r_2(\theta)} f(r,\theta)\,dr\,d\theta$$

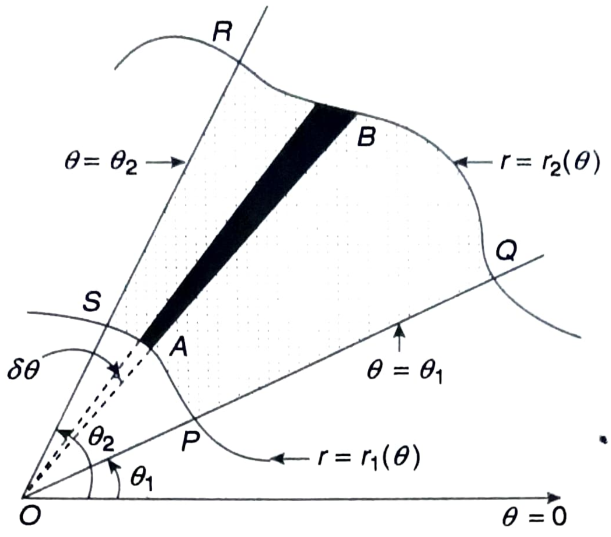

Fig. 5.46

5.4.2 Evaluation of Integrals in Polar Coordinates

Example 1

Evaluate $\int_0^{\frac{\pi}{4}} \int_0^1 r\, dr\, d\theta$.

Solution

$$\begin{aligned}\int_0^{\frac{\pi}{4}} \int_0^1 r\, dr\, d\theta &= \int_0^{\frac{\pi}{4}} \left[\int_0^1 r\, dr\right] d\theta \\ &= \int_0^{\frac{\pi}{4}} \left|\frac{r^2}{2}\right|_0^1 d\theta \\ &= \int_0^{\frac{\pi}{4}} \frac{1}{2}\, d\theta \\ &= \frac{1}{2}\left|\theta\right|_0^{\frac{\pi}{4}} \\ &= \frac{1}{2}\cdot\frac{\pi}{4} \\ &= \frac{\pi}{8}\end{aligned}$$

Another method: Since both the limits are constant and integrand (function) is explicit in r and θ, the integral can be written as

$$\begin{aligned}\int_0^{\frac{\pi}{4}} \int_0^1 r\, dr\, d\theta &= \int_0^{\frac{\pi}{4}} d\theta \cdot \int_0^1 r\, dr \\ &= \left|\theta\right|_0^{\frac{\pi}{4}} \cdot \left|\frac{r^2}{2}\right|_0^1\end{aligned}$$

$$= \frac{\pi}{4} \cdot \frac{1}{2}$$

$$= \frac{\pi}{8}$$

Example 2

Evaluate $\int_0^{\pi} \int_0^{\sin\theta} r\,dr\,d\theta.$

Solution

$$\int_0^{\pi} \int_0^{\sin\theta} r\,dr\,d\theta = \int_0^{\pi} \left[\int_0^{\sin\theta} r\,dr \right] d\theta$$

$$= \int_0^{\pi} \left| \frac{r^2}{2} \right|_0^{\sin\theta} d\theta$$

$$= \frac{1}{2} \int_0^{\pi} \sin^2\theta\,d\theta$$

$$= \frac{1}{2} \int_0^{\pi} \frac{1 - \cos 2\theta}{2}\,d\theta$$

$$= \frac{1}{4} \left| \theta - \frac{\sin 2\theta}{2} \right|_0^{\pi}$$

$$= \frac{1}{4} \left(\pi - \frac{\sin 2\pi}{2} \right)$$

$$= \frac{\pi}{4}$$

Example 3

Evaluate $\int_0^{\frac{\pi}{2}} \int_0^{2a\cos\theta} r^2 \sin\theta\,d\theta\,dr.$

Solution

Since inner limits depend on θ, the function is integrated first w.r.t. r.

The correct form of the integral $= \int_0^{\frac{\pi}{2}} \int_0^{2a\cos\theta} r^2 \sin\theta\,dr\,d\theta$

$$\int_0^{\frac{\pi}{2}} \int_0^{2a\cos\theta} r^2 \sin\theta\,dr\,d\theta = \int_0^{\frac{\pi}{2}} \left[\int_0^{2a\cos\theta} r^2 dr \right] \sin\theta\,d\theta$$

$$= \int_0^{\frac{\pi}{2}} \left| \frac{r^3}{3} \right|_0^{2a\cos\theta} \sin\theta\,d\theta$$

$$= \frac{8a^3}{3} \int_0^{\frac{\pi}{2}} \cos^3\theta \sin\theta\,d\theta$$

$$= -\frac{8a^3}{3}\int_0^{\frac{\pi}{2}} \cos^3\theta(-\sin\theta)\,d\theta$$

$$= -\frac{8a^3}{3}\left|\frac{\cos^4\theta}{4}\right|_0^{\frac{\pi}{2}} \qquad \left[\because \int [f(\theta)]^n f'(\theta)\,d\theta = \frac{[f(\theta)]^{n+1}}{n+1}\right]$$

$$= -\frac{8a^3}{12}\left(\cos^4\frac{\pi}{2} - \cos^4 0\right)$$

$$= \frac{2}{3}a^3$$

Example 4

Evaluate $\int_0^{\frac{\pi}{2}}\int_0^{a(1+\sin\theta)} r^2\cos\theta\,d\theta\,dr.$

Solution

Since inner limits depend on θ, the function is integrated first w.r.t. r.

The correct form of the integral $= \int_0^{\frac{\pi}{2}}\int_0^{a(1+\sin\theta)} r^2\cos\theta\,dr\,d\theta$

$$\int_0^{\frac{\pi}{2}}\int_0^{a(1+\sin\theta)} r^2\cos\theta\,dr\,d\theta = \int_0^{\frac{\pi}{2}}\left[\int_0^{a(1+\sin\theta)} r^2\,dr\right]\cos\theta\,d\theta$$

$$= \int_0^{\frac{\pi}{2}}\left|\frac{r^3}{3}\right|_0^{a(1+\sin\theta)}\cos\theta\,d\theta$$

$$= \frac{1}{3}\int_0^{\frac{\pi}{2}} a^3(1+\sin\theta)^3\cos\theta\,d\theta$$

$$= \frac{a^3}{3}\left|\frac{(1+\sin\theta)^4}{4}\right|_0^{\frac{\pi}{2}} \qquad \left[\because \int [f(\theta)]^n f'(\theta)\,d\theta = \frac{[f(\theta)]^{n+1}}{n+1}\right]$$

$$= \frac{a^3}{12}\left[\left(1+\sin\frac{\pi}{2}\right)^4 - (1+\sin 0)^4\right]$$

$$= \frac{a^3}{12}\left[2^4 - 1\right]$$

$$= \frac{5}{4}a^3$$

Example 5

Evaluate $\int_0^{\frac{\pi}{4}}\int_0^{\sqrt{\cos 2\theta}} \frac{r}{(1+r^2)^2}\,dr\,d\theta.$

Solution

$$\int_0^{\frac{\pi}{4}}\int_0^{\sqrt{\cos 2\theta}} \frac{1}{2}\cdot\frac{2r}{(1+r^2)^2}\,\mathrm{d}r\,\mathrm{d}\theta = \frac{1}{2}\int_0^{\frac{\pi}{4}}\left[\int_0^{\sqrt{\cos 2\theta}}(1+r^2)^{-2}\cdot 2r\mathrm{d}r\right]\mathrm{d}\theta$$

$$= \frac{1}{2}\int_0^{\frac{\pi}{4}}\left|-(1+r^2)^{-1}\right|_0^{\sqrt{\cos 2\theta}}\mathrm{d}\theta \qquad \left[\because \int [f(r)]^n f'(r)\mathrm{d}r = \frac{[f(r)]^{n+1}}{n+1} \quad n \neq -1\right]$$

$$= -\frac{1}{2}\int_0^{\frac{\pi}{4}}\left(\frac{1}{1+\cos 2\theta}-1\right)\mathrm{d}\theta$$

$$= -\frac{1}{2}\int_0^{\frac{\pi}{4}}\left(\frac{1}{2\cos^2\theta}-1\right)\mathrm{d}\theta$$

$$= -\frac{1}{2}\int_0^{\frac{\pi}{4}}\left(\frac{1}{2}\sec^2\theta-1\right)\mathrm{d}\theta$$

$$= -\frac{1}{2}\left|\frac{1}{2}\tan\theta-\theta\right|_0^{\frac{\pi}{4}}$$

$$= -\frac{1}{2}\left(\frac{1}{2}\tan\frac{\pi}{4}-\frac{\pi}{4}\right)$$

$$= \frac{1}{8}(\pi-2)$$

Example 6

Evaluate $\int_0^{\frac{\pi}{2}}\int_0^{\frac{\pi}{2}}\sin(\theta+\phi)\mathrm{d}\theta\,\mathrm{d}\phi$.

Solution

$$\int_0^{\frac{\pi}{2}}\int_0^{\frac{\pi}{2}}\sin(\theta+\phi)\,\mathrm{d}\theta\,\mathrm{d}\phi = \int_0^{\frac{\pi}{2}}\left[\int_0^{\frac{\pi}{2}}\sin(\theta+\phi)\mathrm{d}\theta\right]\mathrm{d}\phi$$

$$= \int_0^{\frac{\pi}{2}}\left|-\cos(\theta+\phi)\right|_0^{\frac{\pi}{2}}\mathrm{d}\phi$$

$$= -\int_0^{\frac{\pi}{2}}\left[\cos\left(\frac{\pi}{2}+\phi\right)-\cos\phi\right]\mathrm{d}\phi$$

$$= -\int_0^{\frac{\pi}{2}}(-\sin\phi-\cos\phi)\mathrm{d}\phi$$

$$= -\left|\cos\phi-\sin\phi\right|_0^{\frac{\pi}{2}}$$

$$= -\left(\cos\frac{\pi}{2}-\sin\frac{\pi}{2}-\cos 0+\sin 0\right)$$

$$= 2$$