$y = x^2,\ x = 1$

Example 16

Evaluate $\iint \frac{dx\,dy}{x^4 + y^2}$, *over the region bounded by the* $y \geq x^2$, $x \geq 1$.

Solution

1. The region of integration is bounded by $y \geq x^2$ (the region inside the parabola $x^2 = y$) and $x \geq 1$ (the region on the right of line $x = 1$).
2. The point of intersection of $x^2 = y$ and $x = 1$ is obtained as $1 = y$.
   The point of intersection is $P(1, 1)$.
3. Here, it is easier to integrate w.r.t. y first than x. Draw a vertical strip AB parallel to y-axis in the region which starts from the parabola $x^2 = y$ and extends up to infinity.
4. Limits of $y : y = x^2$ to $y \to \infty$
   Limits of $x : x = 1$ to $x \to \infty$

$y = x^2$ to $y = \infty$
$x = 1$ to $x = \infty$

$y \to x$
x — const

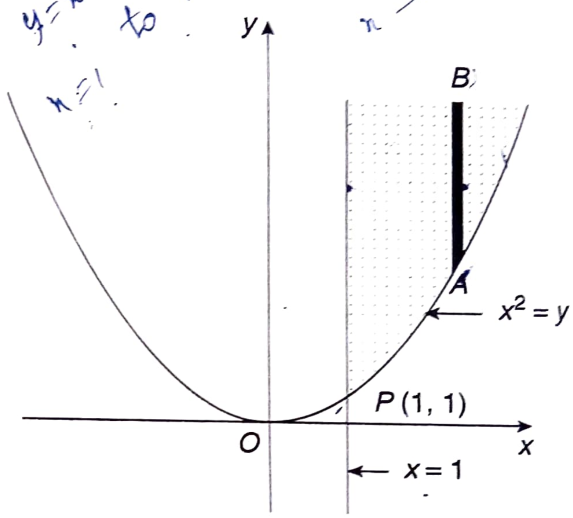


Fig. 5.19

$$I = \int_1^\infty \int_{x^2}^\infty \frac{1}{x^4 + y^2}\,dy\,dx \quad = \int_1^\infty \left(\int_{x^2}^\infty \frac{1}{(x^2)^2 + y^2}\,dy \right) dx$$

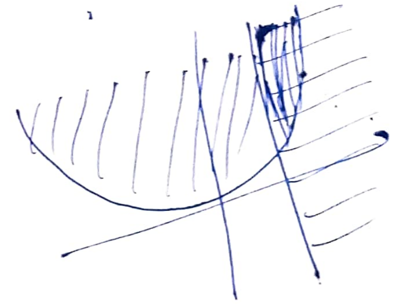
$$= \int_1^\infty \left| \frac{1}{x^2} \tan^{-1} \frac{y}{x^2} \right|_{x^2}^\infty dx \qquad \left(\because \int \frac{1}{a^2 + x^2}\,dx = \frac{1}{a}\tan^{-1}\left(\frac{x}{a}\right)\right)$$

$$= \int_1^\infty \frac{1}{x^2}(\tan^{-1}\infty - \tan^{-1} 1)\,dx$$

$$= \int_1^\infty \frac{1}{x^2}\left(\frac{\pi}{2} - \frac{\pi}{4}\right) dx$$

$$= \frac{\pi}{4}\left| -\frac{1}{x} \right|_1^\infty \qquad \left[\because \int \frac{1}{x^2} = -\frac{1}{x}\right]$$

$$= \frac{\pi}{4}.$$



EXERCISE 5.2

Evaluate the following:

1. $\iint \frac{1}{xy}\,dx\,dy$, over the rectangle $1 \leq x \leq 2$, $1 \leq y \leq 2$.

   [Ans.: $(\log 2)^2$]

2. $\iint \sin \pi(ax + by)\,dx\,dy$, over the triangle bounded by the lines $x = 0$, $y = 0$ and $ax + by = 1$.

   $\left[\text{Ans.: } \frac{1}{\pi ab}\right]$

13

3. $\iint e^{3x+4y}dx\,dy$, over the triangle bounded by the lines $x = 0$, $y = 0$, and $x + y = 1$.

$$\left[\text{Ans.}: \frac{1}{12}(3e^4 - 4e^3 + 1)\right]$$

4. $\iint xy\sqrt{1-x-y}\,dx\,dy$, over the triangle bounded by $x = 0$, $y = 0$ and $x + y = 1$.

$$\left[\text{Ans.}: \frac{16}{945}\right]$$

5. $\iint \sqrt{xy-y^2}\,dx\,dy$, over the triangle having vertices (0, 0), (10, 1), (1, 1).

[Ans.: 6]

6. $\iint (x+y+a)dx\,dy$, over the region bounded by the circle $x^2 + y^2 = a^2$.

[Ans.: πa^3]

7. $\iint xy\,dx\,dy$, over the region bounded by the x-axis, the line $y = 2x$ and the parabola $y = \dfrac{x^2}{4a}$.

$$\left[\text{Ans.}: \frac{2048}{3}a^4\right]$$

8. $\iint 5-2x-y\,(dx)\,dy$, over the region bounded by x-axis, the line $x + 2y = 3$ and the parabola $y^2 = x$.

$$\left[\text{Ans.}: \frac{217}{60}\right]$$

9. $\iint (4x^2-y^2)^{\frac{1}{2}}dx\,dy$, over the triangle bounded by x-axis, the line $y = x$ and $x = 1$.

$$\left[\text{Ans.}: \frac{1}{3}\left(\frac{\pi}{3}+\frac{\sqrt{3}}{2}\right)\right]$$

10. $\iint xy(x+y)dx\,dy$, over the region bounded by the parabola $y^2 = x$, $x^2 = y$.

$$\left[\text{Ans.}: \frac{3}{28}\right]$$

11. $\iint xy(x+y)dx\,dy$, over the region bounded by the curve $x^2 = y$ and the line $x = y$.

$$\left[\text{Ans.}: \frac{3}{56}\right]$$

12. $\iint xy(x-1)dx\,dy$, over the region bounded by the rectangular hyperbola $xy = 4$, the lines $y = 0$, $x = 1$, $x = 4$ and x-axis.

[Ans.: 8(3 – log 4)]

5.3 CHANGE OF ORDER OF INTEGRATION

Sometimes, evaluation of double integral becomes easier by changing the order of integration. To change the order of integration, first, we draw the region of integration with the help of the given limits. Then we draw a vertical or horizontal strip as per the required order of integration. This change of order also changes the limits of integration.

5.3.1 Changing the Order of Integration

Example 1

Change the order of integration of $\int_0^1 \int_0^x f(x, y)\mathrm{d}x\,\mathrm{d}y$.

Solution

1. Since inner limits depends on x, the function is integrated first w.r.t. y and then w.r.t. x.

   The correct form of the integral

   $$= \int_0^1 \int_0^x f(x, y)\mathrm{d}y\,\mathrm{d}x.$$

2. Limits of y : $y = 0$ to $y = x$, along vertical strip $A'B'$

   Limits of x : $x = 0$ to $x = 1$
3. The region is bounded by the lines $y = 0$, $y = x$, and $x = 1$.
4. The point of intersection of $y = x$ and $x = 1$ is $Q(1, 1)$.
5. To change the order of integration, i.e., to integrate first w.r.t. x, draw a horizontal strip AB parallel to x-axis which starts from the line $y = x$ and terminates on the line $x = 1$.

   Limits of x : $x = y$ to $x = 1$

   Limits of y : $y = 0$ to $y = 1$

Hence, the given integral after change of order is

$$\int_0^1 \int_0^x f(x, y)\mathrm{d}y\,\mathrm{d}x = \int_0^1 \int_y^1 f(x, y)\mathrm{d}x\,\mathrm{d}y$$

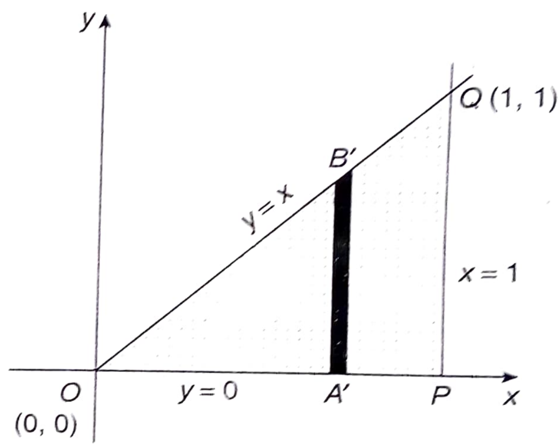


Fig. 5.20(a)

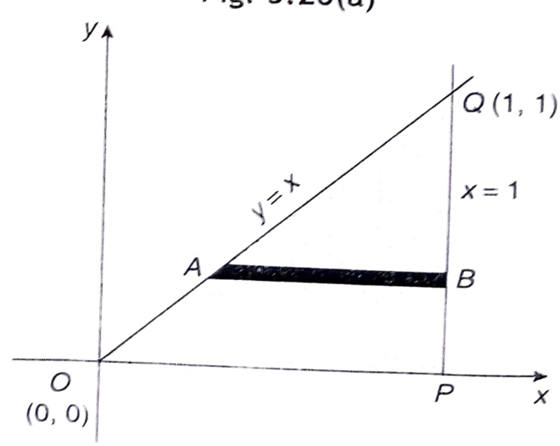


Fig. 5.20(b)

Example 2

Change the order of integration of $\int_0^1 \int_0^y f(x, y)\mathrm{d}y\,\mathrm{d}x$.

Solution

1. Since inner limits depend on y, the function is integrated first w.r.t. x and then w.r.t. y. The correct form of the integral
$$= \int_0^1 \int_0^y f(x, y) \mathrm{d}x \, \mathrm{d}y.$$
2. Limits of $x : x = 0$ to $x = y$, along horizontal strip $A'B'$
Limits of $y : y = 0$ to $y = 1$
3. The region is bounded by the lines $x = 0$, $x = y$, and $y = 1$.
4. The point of intersection of $x = y$ and $y = 1$ is $Q\ (1, 1)$.
5. To change the order of integration, i.e., to integrate first w.r.t. y, draw a vertical strip AB parallel to y-axis which starts from the line $x = y$ and terminates on the line $y = 1$.
Limits of $y : y = x$ to $y = 1$
Limits of $x : x = 0$ to $x = 1$

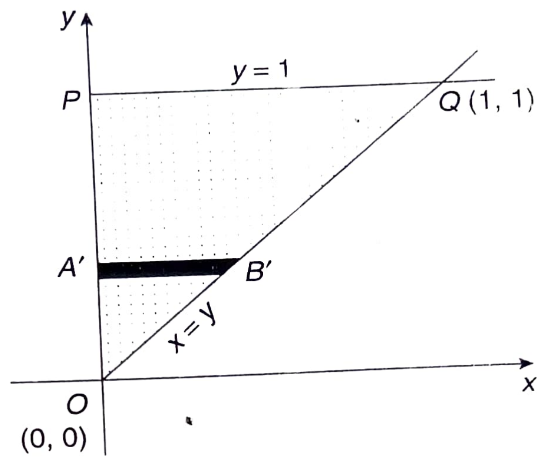


Fig. 5.21(a)

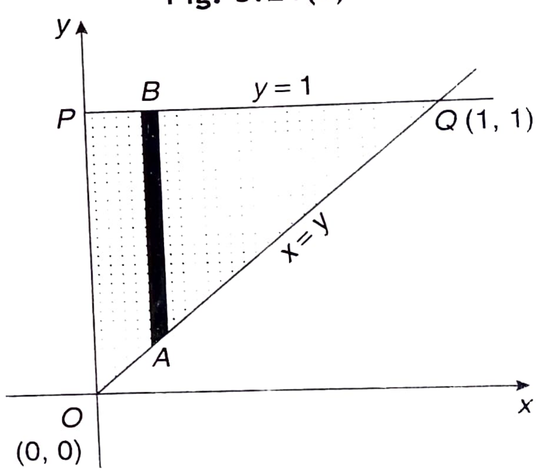


Fig. 5.21(b)

Hence, the given integral after change of order is

$$\int_0^1 \int_0^y f(x, y) \mathrm{d}x \, \mathrm{d}y = \int_0^1 \int_x^1 f(x, y) \mathrm{d}y \, \mathrm{d}x$$

Example 3

Change the order of integration of $\int_0^a \int_x^a f(x, y) \mathrm{d}y \, \mathrm{d}x$.

Solution

1. Since inner limits depend on x, the function is integrated first w.r.t. y and then w.r.t. x.
2. Limits of $y : y = x$ to $y = a$, along vertical strip $A'B'$.
Limits of $x : x = 0$ to $x = a$
3. The region is bounded by the lines $y = x$, $y = a$ and $x = 0$.
4. The point of intersection of $y = x$ and $y = a$ is $Q\ (a, a)$.
5. To change the order of integration i.e., to integrate first w.r.t. x, draw a horizontal strip AB parallel to x-axis which starts from

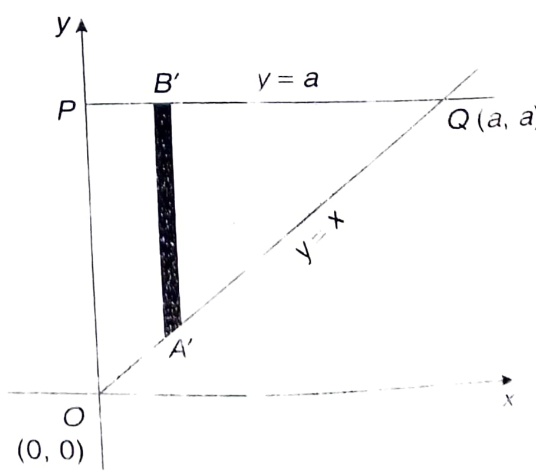


Fig. 5.22(a)

the line $x = 0$ and terminates on the line $y = x$.

Limits of $x : x = 0$ to $x = y$

Limits of $y : y = 0$ to $y = a$

Hence, the given integral after change of order is

$$\int_0^a \int_x^a f(x, y)\mathrm{d}y\,\mathrm{d}x = \int_0^a \int_0^y f(x, y)\mathrm{d}x\,\mathrm{d}y$$

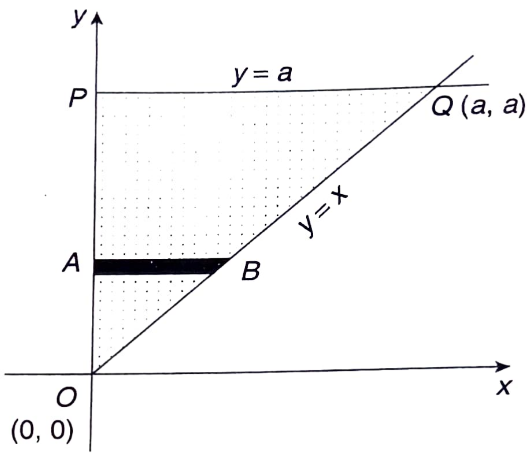


Fig. 5.22(b)

Example 4

Change the order of integration of $\int_0^\infty \int_x^\infty f(x, y)\mathrm{d}x\,\mathrm{d}y$.

Solution

1. Since inner limits depend on x, the function is integrated first w.r.t. y and then w.r.t. x.
   The correct form of the integral $= \int_0^\infty \int_x^\infty f(x, y)\mathrm{d}y\,\mathrm{d}x$

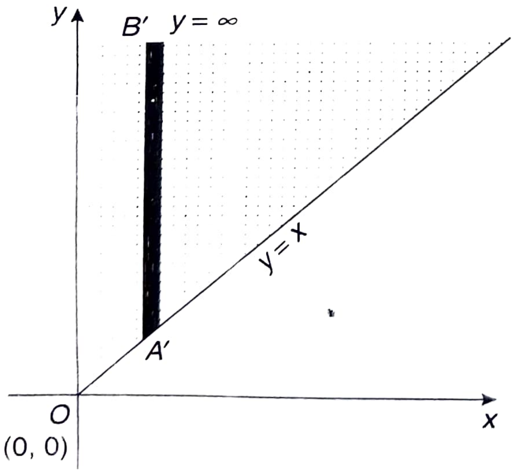


Fig. 5.23(a)

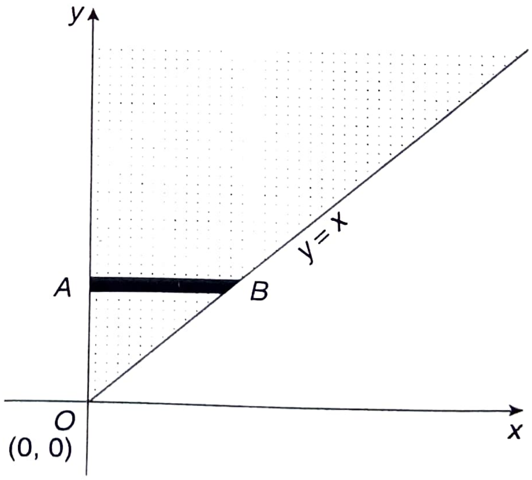


Fig. 5.23(b)

2. Limits of $y : y = x$ to $y = \infty$, along vertical strip
   Limits of $x : x = 0$ to $x = \infty$
3. The region is bounded by the lines $y = x$ and $x = 0$.
4. Here the only point of intersection is origin O.
5. To change the order of integration, i.e., to integrate first w.r.t. x, draw a horizontal strip AB parallel to x-axis which starts from the line $x = 0$ and terminates on the line $y = x$.
   Limits of $x :$ $x = 0$ to $x = y$

Limits of y : $y = 0$ to $y = \infty$

Hence, the given integral after change of order is

$$\int_0^\infty \int_x^\infty f(x, y)\,dy\,dx = \int_0^\infty \int_0^y f(x, y)\,dx\,dy.$$

Example 5

Change the order of integration of $\int_0^1 \int_x^{\sqrt{x}} f(x,y)\,dy\,dx.$

Solution

1. The function is integrated first w.r.t. y and then w.r.t. x.
2. Limits of y : $y = x$ to $y = \sqrt{x}$

   Limits of x : $x = 0$ to $x = 1$
3. The region is bounded by the line $y = x$ and the parabola $y^2 = x$.
4. The points of intersection of $y^2 = x$ and $y = x$ are obtained as

$$x^2 = x$$
$$x = 0, 1$$
$$\therefore\ y = 0, 1.$$

   The points of intersection are $O\,(0, 0)$ and $Q\,(1, 1)$.
5. To change the order of integration, i.e., to integrate first w.r.t. x, draw a horizontal strip AB parallel to x-axis which starts from the parabola $y^2 = x$ and terminates on the line $y = x$.

   Limits of x : $x = y^2$ to $x = y$

   Limits of y : $y = 0$ to $y = 1$

   Hence, the given integral after change of order is

$$\int_0^1 \int_x^{\sqrt{x}} f(x, y)\,dy\,dx = \int_0^1 \int_{y^2}^{y} f(x, y)\,dx\,dy$$

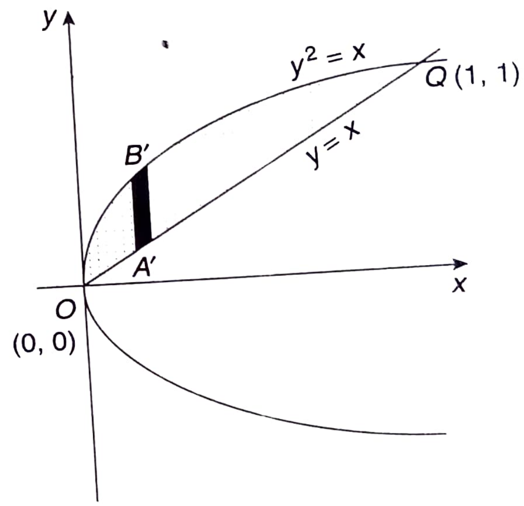


Fig. 5.24(a)

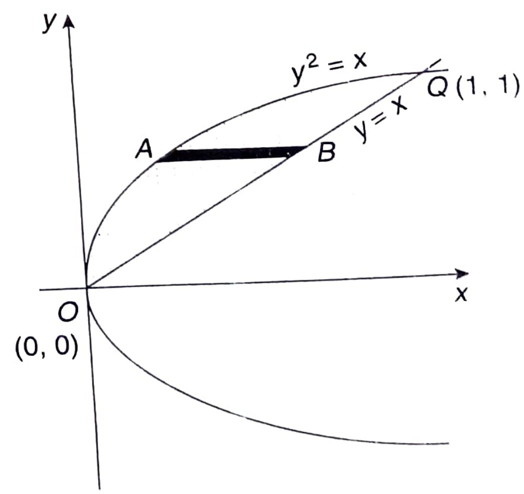


Fig. 5.24(b)

Example 6

Change the order of integration of $\int_0^1 \int_{y^2}^{y^{\frac{1}{3}}} f(x, y)\,dx\,dy.$

Solution

1. The function is integrated first w.r.t. x and then w.r.t. y.
2. Limits of x : $x = y^2$ to $x = y^{\frac{1}{3}}$

   Limits of y : $y = 0$ to $y = 1$
3. The region is bounded by the parabola $y^2 = x$ and the cubical parabola $y = x^3$.
4. The points of intersection of $y^2 = x$ and $y = x^3$ are obtained as

$$x^6 = x$$
$$x = 0, 1$$
$$\therefore y = 0, 1.$$

   The points of intersection are O (0, 0) and Q (1, 1).

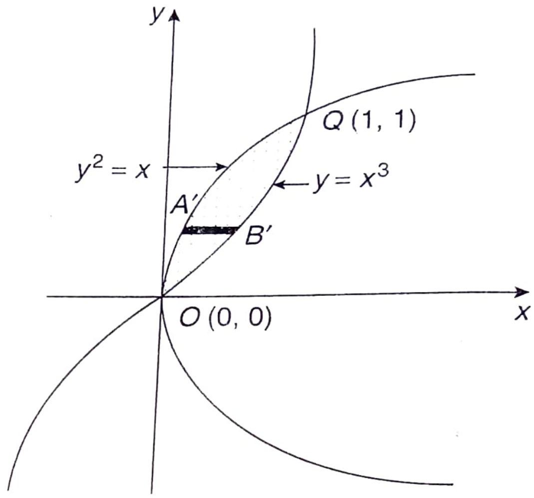


Fig. 5.25(a)

5. To change the order of integration, i.e., to integrate first w.r.t. y, draw a vertical strip parallel to y-axis which starts from the cubical parabola $y = x^3$ and terminates on the parabola $y^2 = x$.

   Limits of y : $y = x^3$ to $y = \sqrt{x}$

   Limits of x : $x = 0$ to $x = 1$

Hence, the given integral after change of order is

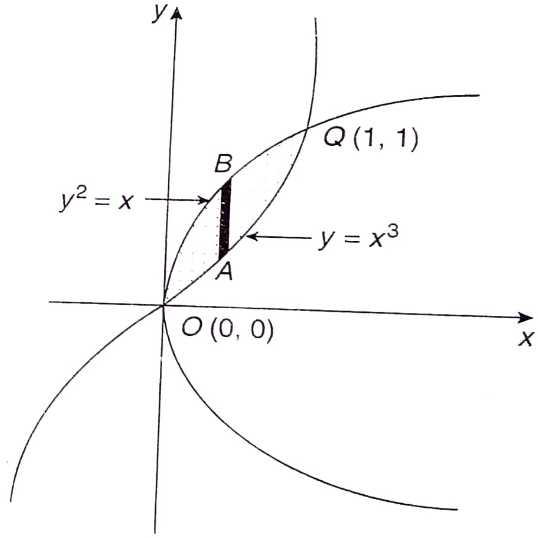


Fig. 5.25(b)

$$\int_0^1 \int_{y^2}^{y^{\frac{1}{3}}} f(x, y)\mathrm{d}x\,\mathrm{d}y = \int_0^1 \int_{x^3}^{\sqrt{x}} f(x, y)\mathrm{d}y\,\mathrm{d}x$$

Example 7

Change the order of integration of $\int_0^8 \int_{\frac{y-8}{4}}^{\frac{y}{4}} f(x, y)\mathrm{d}x\,\mathrm{d}y$.

Solution

1. The function is integrated first w.r.t. x and then w.r.t. y.
2. Limits of x : $x = \dfrac{y-8}{4}$ to $x = \dfrac{y}{4}$

   Limits of y : $y = 0$ to $y = 8$
3. The region is bounded by the line $y = 4x + 8$, $y = 4x$, $y = 8$ and x-axis ($y = 0$).

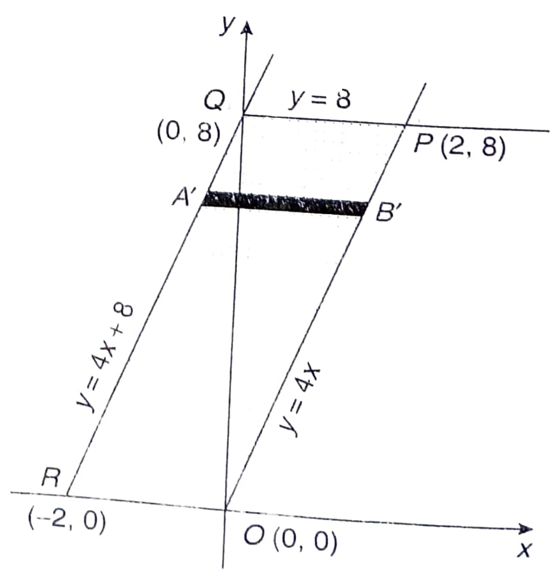


Fig. 5.26(a)

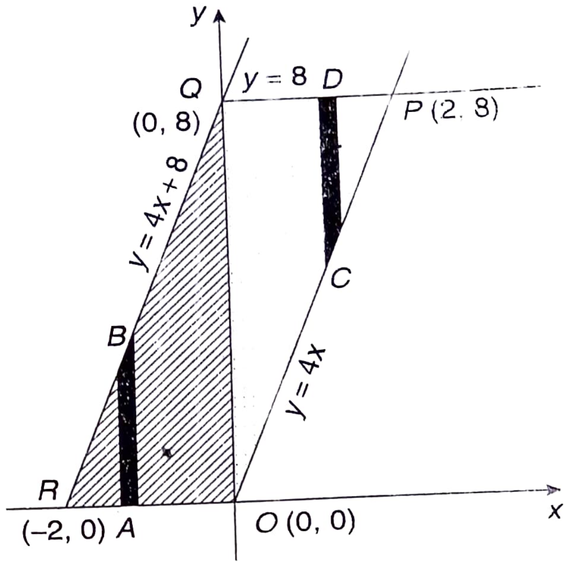


Fig. 5.26(b)

4. The point of intersection of $y = 4x$ and $y = 8$ is obtained as
$$8 = 4x$$
$$x = 2.$$
The point of intersection is P (2, 8).
5. To change the order of integration, i.e., to integrate first w.r.t. y, divide the region $OPQR$ into two subregions OQR and OPQ. Draw a vertical strip parallel to y-axis in each subregion.
   (i) In subregion OQR, strip AB starts from x-axis and terminates on the line $y = 4x + 8$.
   Limits of y : $y = 0$ to $y = 4x + 8$
   Limits of x : $x = -2$ to $x = 0$
   (ii) In subregion OPQ, strip CD starts from the line $y = 4x$ and terminates on the line $y = 8$.
   Limits of y : $y = 4x$ to $y = 8$
   Limits of x : $x = 0$ to $x = 2$

Hence, the given integral after change of order is
$$\int_0^8 \int_{\frac{y-8}{4}}^{\frac{y}{4}} f(x, y)\,dx\,dy = \int_{-2}^{0} \int_0^{4x+8} f(x, y)\,dy\,dx + \int_0^2 \int_{4x}^{8} f(x, y)\,dy\,dx$$

Example 8

Change the order of integration of $\int_{-a}^{a} \int_0^{\frac{y^2}{a}} f(x, y)\,dx\,dy$.

Solution

1. The function is integrated first w.r.t. x and then w.r.t. y.
2. Limits of x : $x = 0$ to $x = \frac{y^2}{a}$
   Limits of y : $y = -a$ to $y = a$
3. The region is bounded by the y-axis, the parabola $y^2 = ax$, and the line $y = -a$, and $y = a$.
4. (i) The point of intersection of $y^2 = ax$ and $y = -a$ is obtained as
$$a^2 = ax$$
$$x = a$$
The point of intersection is R $(a, -a)$.

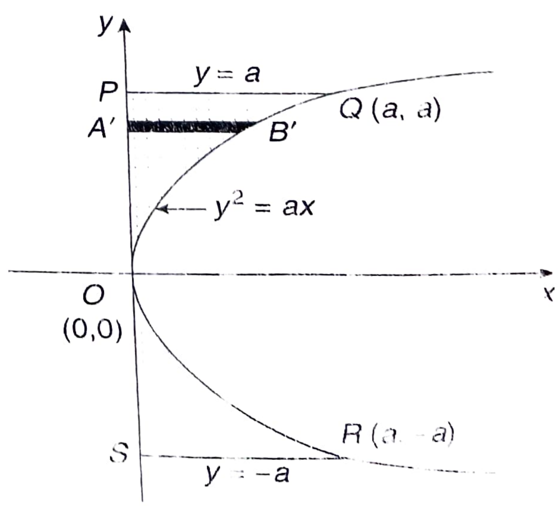


Fig. 5.27(a)

(ii) The point of intersection of $y^2 = ax$ and $y = a$ is obtained as

$$a^2 = ax$$
$$x = a$$

The point of intersection is $Q\,(a, a)$.

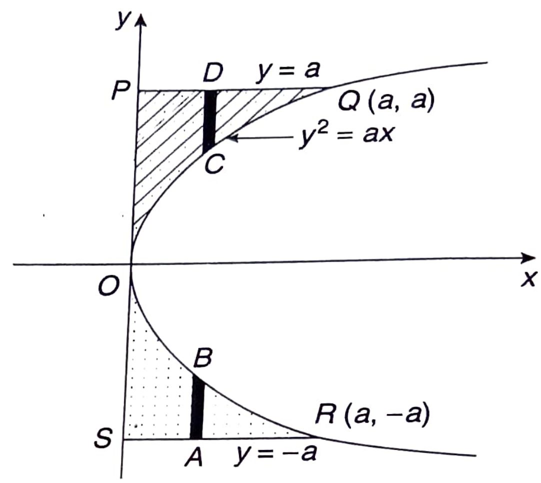


Fig. 5.27(b)

5. To change the order of integration, i.e., to integrate first w.r.t. y, divide the region into two subregions ORS and OPQ. Draw a vertical strip parallel to y-axis in each subregion.

   (i) In subregion ORS, strip AB starts from the line $y = -a$ and terminates on the parabola $y^2 = ax$.
   Limits of y : $y = -a$ to $y = -\sqrt{ax}$ (part of the parabola below x-axis)
   Limits of x : $x = 0$ to $x = a$

   (ii) In subregion OPQ, strip CD starts from the parabola $y^2 = ax$ and terminates on the line $y = a$.
   Limits of y : $y = \sqrt{ax}$ to $y = a$
   Limits of x : $x = 0$ to $x = a$

Hence, the given integral after change of order is

$$\int_{-a}^{a}\int_{0}^{\frac{y^2}{a}} f(x, y)\,dx\,dy = \int_{0}^{a}\int_{-a}^{-\sqrt{ax}} f(x, y)\,dy\,dx + \int_{0}^{a}\int_{\sqrt{ax}}^{a} f(x, y)\,dy\,dx$$

Example 9

Change the order of integration of $\displaystyle\int_0^2\int_y^{2+\sqrt{4-2y}} f(x, y)\,dx\,dy.$

Solution

1. The function is integrated first w.r.t. x and then w.r.t. y.
2. Limits of x : $x = y$ to $x = 2+\sqrt{4-2y}$
   Limits of y : $y = 0$ to $y = 2$
3. The region is bounded by the x-axis, the line $y = x$ and the parabola $(x-2)^2 = 2(2-y)$.
4. The points of intersection of $y = x$ and $(x-2)^2 = 2(2-y)$ are obtained as

$$(x-2)^2 = 2(2-x)$$
$$x = 0, 2$$
$$\therefore\ y = 0, 2.$$

The points of intersection are $O\,(0, 0)$ and $Q\,(2, 2)$.

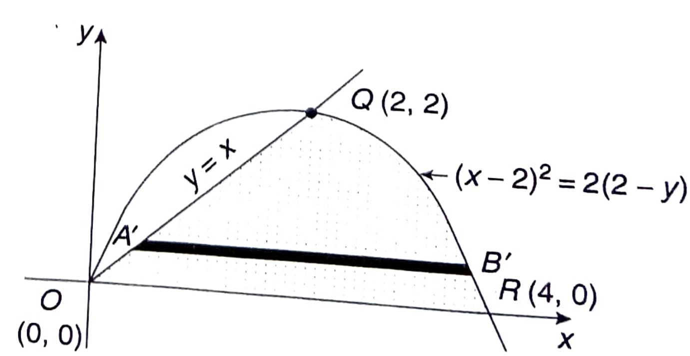


Fig. 5.28(a)

5. To change the order of integration, i.e., to integrate first w.r.t. y, divide the region into two subregions OPQ and PQR. Draw a vertical strip parallel to y-axis in each subregion.

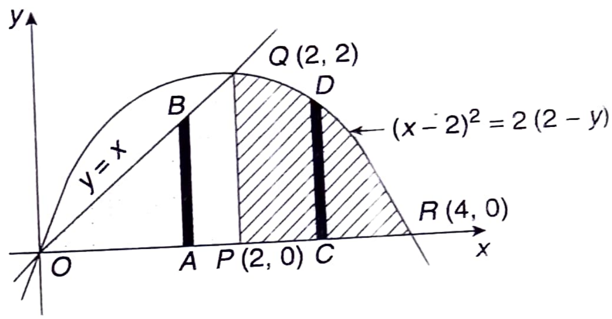


Fig. 5.28(b)

(i) In subregion OPQ, strip AB starts from x-axis and terminates on the line $y = x$.
Limits of y : $y = 0$ to $y = x$
Limits of x : $x = 0$ to $x = 2$

(ii) In subregion PQR, strip CD starts from x-axis and terminates on the parabola $(x-2)^2 = 2(2-y)$.

Limits of y : $y = 0$ to $y = 2x - \dfrac{x^2}{2}$

Limits of x : $x = 2$ to $x = 4$

Hence, the given integral after change of order is

$$\int_0^2 \int_y^{2+\sqrt{4-2y}} f(x, y)\,dx\,dy = \int_0^2 \int_0^x f(x, y)\,dy\,dx + \int_2^4 \int_0^{2x-\frac{x^2}{2}} f(x, y)\,dy\,dx$$

Example 10

Change the order of integration of $\displaystyle\int_0^{a\cos\alpha} \int_{x\tan\alpha}^{\sqrt{a^2-x^2}} f(x, y)\,dy\,dx.$

Solution

1. The function is integrated first w.r.t. y and then w.r.t. x.
2. Limits of y : $y = x \tan \alpha$ to $y = \sqrt{a^2 - x^2}$
   Limits of x : $x = 0$ to $x = a \cos \alpha$
3. The region is bounded by the line $y = x \tan \alpha$, the circle $x^2 + y^2 = a^2$ and y-axis. Since given limits of x and y are positive, the region lies in the first quadrant.
4. The points of intersection of $y = x \tan \alpha$ and $x^2 + y^2 = a^2$ are obtained as

$$x^2 + x^2 \tan^2 \alpha = a^2$$
$$x = \pm a \cos \alpha$$
$$\therefore\ y = \pm a \sin \alpha.$$

   The points of intersection are $P\,(a \cos \alpha, a \sin \alpha)$ and $P'(-a \cos \alpha, -a \sin \alpha)$.

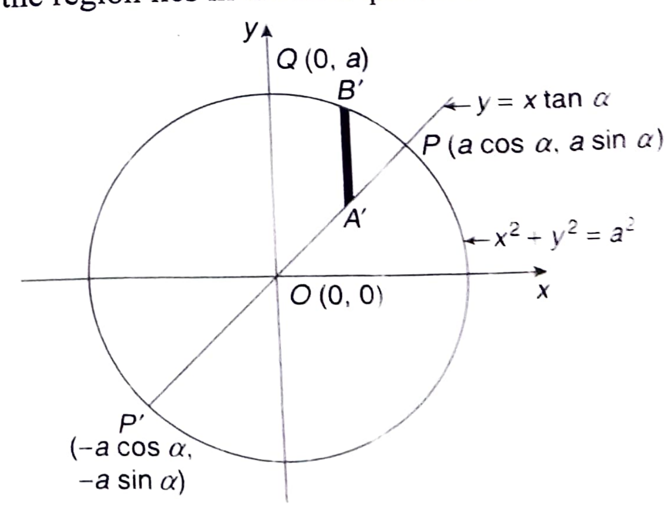


Fig. 5.29(a)

5. To change the order of integration, i.e., to integrate first w.r.t. x, divide the region into two subregions OPR and PQR. Draw a horizontal strip in each subregion.

(i) In subregion OPR, strip AB starts from y-axis and terminates on the line $y = x \tan\alpha$.
Limits of x : $x = 0$ to $x = y\cot\alpha$
Limits of y : $y = 0$ to $y = a\sin\alpha$

(ii) In subregion PQR, strip CD starts from y-axis and terminates on the circle $x^2 + y^2 = a^2$.
Limits of x : $x = 0$ to $x = \sqrt{a^2 - y^2}$
Limits of y : $y = a\sin\alpha$ to $y = a$

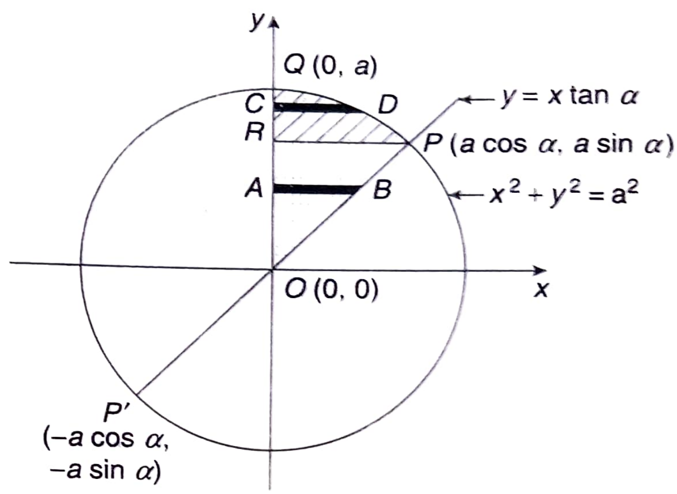


Fig. 5.29(b)

Hence, given integral after change of order is

$$\int_0^{a\cos\alpha}\int_{x\tan\alpha}^{\sqrt{a^2-x^2}} f(x, y)\,dy\,dx = \int_0^{a\sin\alpha}\int_0^{y\cot\alpha} f(x, y)\,dx\,dy$$
$$+\int_{a\sin\alpha}^{a}\int_0^{\sqrt{a^2-y^2}} f(x, y)\,dx\,dy$$

Example 11

Change the order of integration of $\int_0^4\int_{\sqrt{4x-x^2}}^{\sqrt{4x}} f(x, y)\,dy\,dx$.

Solution

1. The function is integrated first w.r.t. y and then w.r.t. x.
2. Limits of y : $y = \sqrt{4x - x^2}$ to $y = \sqrt{4x}$
   Limits of x : $x = 0$ to $x = 4$.
3. The region is bounded by the circle $x^2 + y^2 - 4x = 0$, the parabola $y^2 = 4x$ and the line $x = 4$.
4. (i) The point of intersection of $x^2 + y^2 - 4x = 0$ and $y^2 = 4x$ is obtained as

$$x^2 = 0$$
$$x = 0$$
$$\therefore y = 0.$$

   The point of intersection is O (0, 0).

   (ii) The points of intersection of $y^2 = 4x$ and $x = 4$ are obtained as

$$y^2 = 16$$
$$y = \pm 4$$

   The points of intersection are Q (4, 4) and Q' (4, −4).

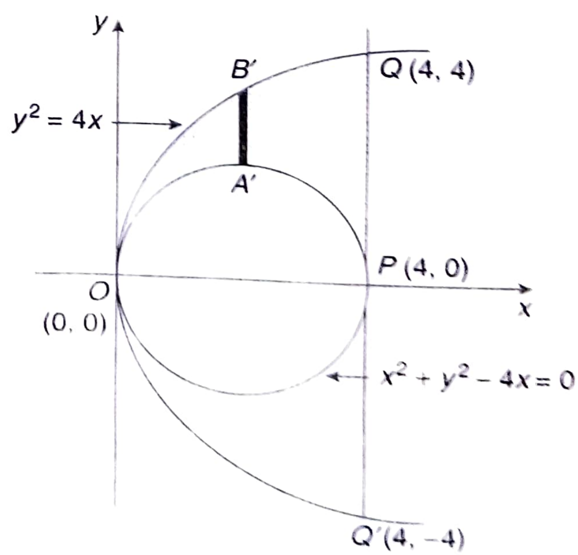


Fig. 5.30(a)

5. To change the order of integration, i.e., to integrate first w.r.t. x, divide the region into three subregions ORT, TPS and RSQ. Draw a horizontal strip parallel to x-axis in each subregion.

   (i) In subregion ORT, strip AB starts from the parabola $y^2 = 4x$ and terminates on the circle $x^2 + y^2 - 4x = 0$.
   Limits of x :
   $$x = \frac{y^2}{4} \text{ to } x = 2 - \sqrt{4 - y^2}$$
   (Part of the circle where $x < 2$)
   Limits of y : $y = 0$ to $y = 2$

   (ii) In subregion TPS, strip CD starts from the circle $x^2 + y^2 - 4x = 0$ and terminates on the line $x = 4$.
   Limits of x : $x = 2 + \sqrt{4 - y^2}$ (Part of circle where $x > 2$) to $x = 4$
   Limits of y : $y = 0$ to $y = 2$

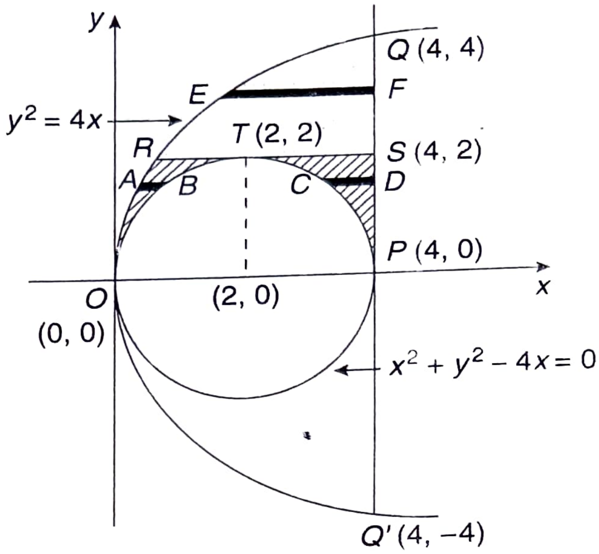


Fig. 5.30(b)

   (iii) In subregion RSQ, strip EF starts from the parabola $y^2 = 4x$ and terminates on the line $x = 4$.
   Limits of x : $x = \frac{y^2}{4}$ to $x = 4$
   Limits of y : $y = 2$ to $y = 4$

Hence, given integral after change of order is

$$\int_0^4 \int_{\sqrt{4x - x^2}}^{\sqrt{4x}} f(x, y)\,dy\,dx = \int_0^2 \int_{\frac{y^2}{4}}^{2-\sqrt{4-y^2}} f(x, y)\,dx\,dy + \int_0^2 \int_{2+\sqrt{4-y^2}}^{4} f(x, y)\,dx\,dy + \int_2^4 \int_{\frac{y^2}{4}}^{4} f(x, y)\,dx\,dy$$

Example 12

Change the order of integration of $\int_0^2 \int_{\sqrt{4-x}}^{(4-x)^2} f(x, y)\,dy\,dx$.

Solution

1. The function is integrated first w.r.t. y and then w.r.t. x.
2. Limits of y : $y = \sqrt{4 - x}$ to $y = (4 - x)^2$
   Limits of x : $x = 0$ to $x = 2$
3. The region is enclosed by the parabolas $y^2 = 4 - x$, $y = (4 - x)^2$, the lines $x = 0$ and $x = 2$.
4. (i) The points of intersection of $x = 2$ and $y^2 = 4 - x$ are obtained as

$$y^2 = (4 - 2)$$
$$y = \pm\sqrt{2}.$$

The points of intersection are $Q\,(2, \sqrt{2})$ and $Q'(2, -\sqrt{2})$.

(ii) The point of intersection of $x = 2$ and $y = (4 - x)^2$ is obtained as
$$y = (4 - 2)^2 = 4.$$
The point of intersection is $S\,(2, 4)$.

(iii) The points of intersection of $x = 0$ and $y^2 = 4 - x$ are obtained as
$$y^2 = 4$$
$$y = \pm 2.$$
The points of intersection are $P\,(0, 2)$ and $P'\,(0, -2)$.

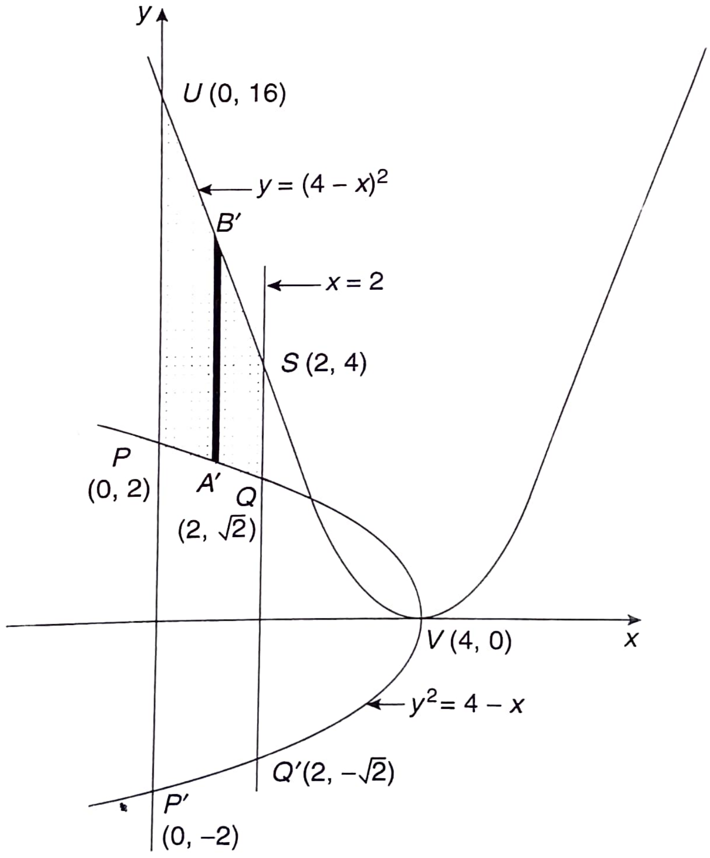


Fig. 5.31(a)

(iv) The point of intersection of $x = 0$ and $y = (4 - x)^2$ is obtained as
$y = 16.$
The point of intersection is $U\,(0, 16)$.

5. To change the order of integration, i.e., to integrate first w.r.t. x, divide the region into three subregions PQR, $PRST$ and STU. Draw a horizontal strip in each subregion.

(i) In subregion PQR, strip AB starts from the parabola $y^2 = 4 - x$ and terminates on the line $x = 2$.

Limits of x : $x = 4 - y^2$ to $x = 2$

Limits of y : $y = \sqrt{2}$ to $y = 2$

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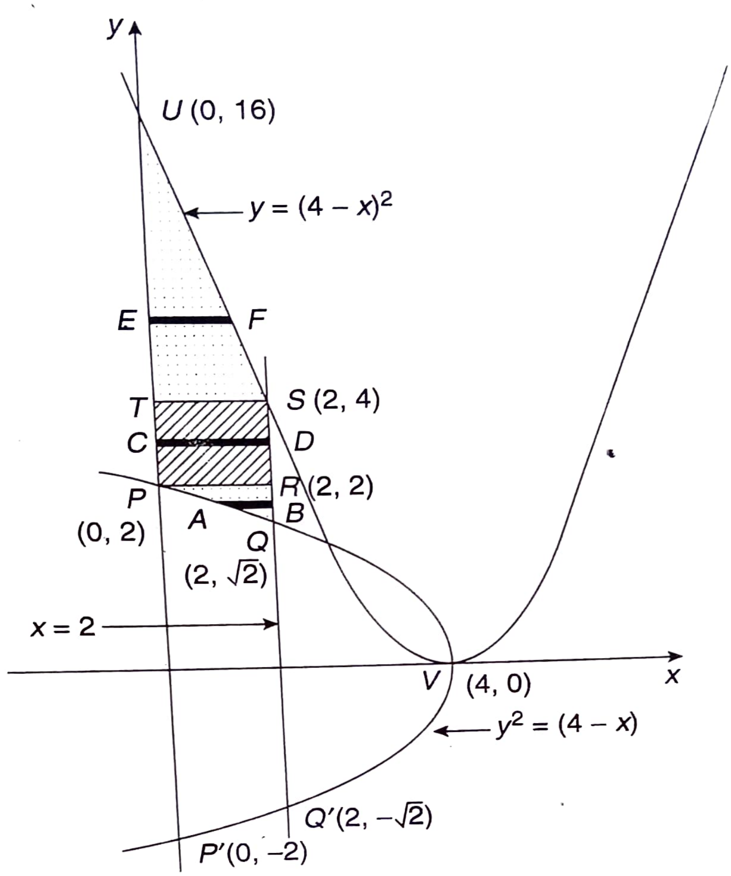

Fig. 5.31(b)

(ii) In subregion $PRST$, strip CD starts from y-axis and terminates on the line $x = 2$.
Limits of x : $x = 0$ to $x = 2$
Limits of y : $y = 2$ to $y = 4$

(iii) In subregion STU, strip EF starts from y-axis and terminates on the parabola $y = (4 - x)^2$.
Limits of x : $x = 0$ to $x = 4 - \sqrt{y}$ (Part of the parabola where $x < 4$)
Limits of y : $y = 4$ to $y = 16$

Hence, the given integral after change of order is

$$\int_0^2 \int_{\sqrt{4-x}}^{(4-x)^2} f(x, y)\,dy\,dx = \int_{\sqrt{2}}^{2} \int_{4-y^2}^{2} f(x, y)\,dx\,dy + \int_2^4 \int_0^2 f(x, y)\,dx\,dy$$
$$+ \int_4^{16} \int_0^{4-\sqrt{y}} f(x, y)\,dx\,dy$$

5.3.2 Change the Order of Integration and Evaluate the Following

Example 1

Change the order of integration and evaluate $\int_0^a \int_x^a (x^2 + y^2)\,dy\,dx$.

(y = π , y = x)

Solution

1. Since inner limits depend on x, the function is integrated first w.r.t. y.
2. Limits of y : $y = x$ to $y = a$, along vertical strip
   Limits of x : $x = 0$ to $x = a$
3. The region is bounded by the lines $y = x$, $y = a$ and $x = 0$
4. The point of intersection of $y = x$ and $y = a$ is $Q\ (a, a)$
5. To change the order of integration, i.e. to integrate first w.r.t. x, draw a horizontal strip AB parallel to x-axis which starts from y-axis and terminates on the line $y = x$.
   Limits of x : $x = 0$ to $x = y$
   Limits of y : $y = 0$ to $y = a$

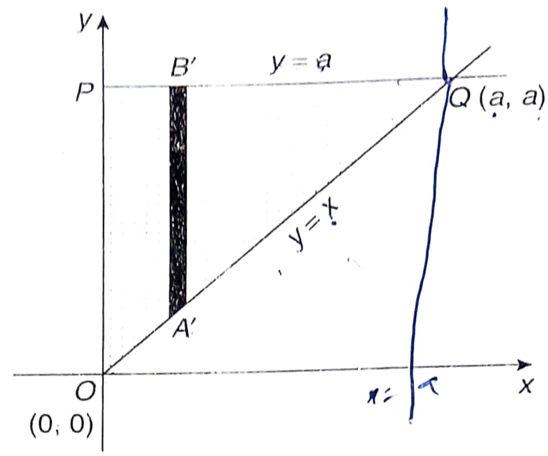

Fig. 5.32(a)

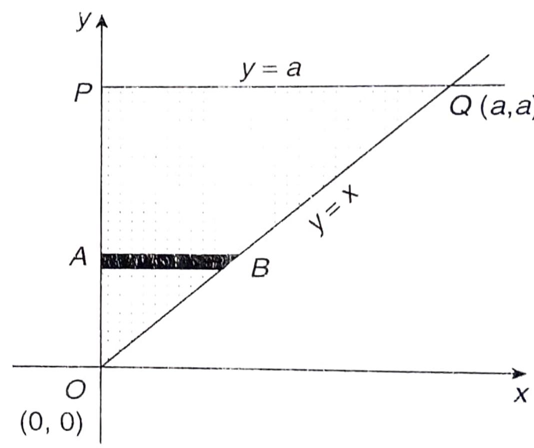

Fig. 5.32(b)

Hence, the given integral after change of order is

$$\int_0^a\int_x^a (x^2+y^2)\,dy\,dx = \int_0^a\int_0^y (x^2+y^2)\,dx\,dy$$

$$= \int_0^a \left|\frac{x^3}{3} + y^2 x\right|_0^y dy$$

$$= \int_0^a \left(\frac{y^3}{3} + y^3\right) dy$$

$$= \int_0^a \frac{4}{3} y^3\, dy$$

$$= \frac{4}{3}\left|\frac{y^4}{4}\right|_0^a$$

$$= \frac{a^4}{3}$$

Example 2

Change the order of integration and evaluate $\displaystyle\int_0^{\pi}\int_x^{\pi} \frac{\sin y}{y}\,dy\,dx$.

Solution

1. The function is integrated first w.r.t. y, but evaluation becomes easier by changing the order of integration.
2. Limits of y : $y = x$ to $y = \pi$
   Limits of x : $x = 0$ to $x = \pi$