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# CHAPTER 5

# Multiple Integrals

## 5.1 INTRODUCTION

Integration of functions of two or more variables is normally called multiple integration. The particular case of integration of functions of two variables is called *double integration* and that of three variables is called *triple integration*. Sometimes, we have to change the variables to simplify the integrand while evaluating the multiple integrals. Variables can be changed by substitution or by changing the coordinate system (polar, spherical or cylindrical coordinates). Multiple integrals are useful in evaluating plane area, mass of a lamina, mass and volume of solid regions, etc.

## 5.2 DOUBLE INTEGRALS

Let $f(x, y)$ be a continuous function defined in a closed and bounded region $R$ in the $xy$-plane. Divide the region $R$ into small elementary rectangles by drawing lines parallel to coordinate axes. Let the total number of complete rectangles which lie inside the region $R$ is $n$. Let $\delta A_r$ be the area of $r^{\text{th}}$ rectangle and $(x_r, y_r)$ be any point in this rectangle.

Consider the sum

$$S = \sum_{r=1}^{n} f(x_r, y_r)\delta A_r \qquad \ldots(1)$$

where $\delta A_r = \delta x_r \cdot \delta y_r$.

If we increase the number of elementary rectangles then the area of each rectangle decreases. Hence, as $n \to \infty$, $\delta A_r \to 0$. The limit of the sum given by the Eq. (1), if it exists, is called the double integral of $f(x, y)$ over the region $R$ and is denoted by $\iint_R f(x, y)\,\mathrm{d}A$.

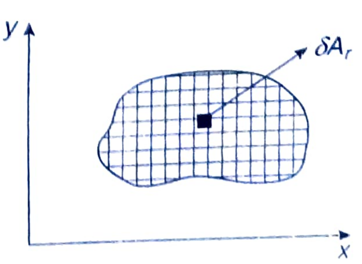


Fig. 5.1

Hence,

$$\iint_R f(x, y)\,\mathrm{d}A = \lim_{\substack{n\to\infty \\ \delta A_r \to 0}} \sum_{r=1}^{n} f(x_r, y_r)\delta A_r$$

where $\mathrm{d}A = \mathrm{d}x\,\mathrm{d}y$

## Evaluation of Double Integrals

Double integral of a function $f(x, y)$ over a region $R$ can be evaluated by two successive integrations. There are two different methods to evaluate a double integral.

***Method-I*** Let the region $R$, i.e., $PQRS$ be bounded by the curves $y = y_1(x)$, $y = y_2(x)$ and the lines $x = a$, $x = b$.

In the region $PQRS$, draw a vertical strip $AB$. Along the strip $AB$, $y$ varies from $y_1$ to $y_2$ and $x$ is fixed. Therefore, the double integral is integrated first w.r.t. $y$ between the limits $y_1$ and $y_2$ treating $x$ as constant.

Now, move the strip $AB$ horizontally from $PS$ (i.e., $x = a$) to $QR$ (i.e., $x = b$) to cover the entire region $PQRS$. The result of the first integral is integrated w.r.t. $x$ between the limits $a$ and $b$.

Hence,

$$\iint_R f(x, y)\,dx\,dy = \int_a^b \left[\int_{y_1(x)}^{y_2(x)} f(x, y)\,dy\right] dx$$

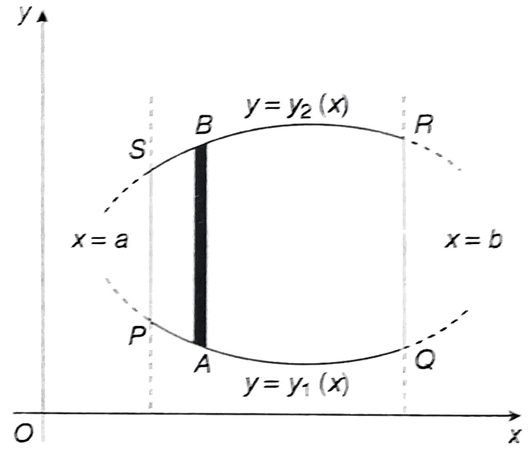


Fig. 5.2

***Method-II*** Let the region $R$ be bounded by the curves $x = x_1(y)$, $x = x_2(y)$ and the lines $y = c$, $y = d$.

In the region $PQRS$, draw a horizontal strip $AB$. Along the strip $AB$, $x$ varies from $x_1$ to $x_2$ and $y$ is fixed. Therefore, the double integral is integrated first w.r.t. $x$ between the limits $x_1$ and $x_2$ treating $y$ as constant.

Now, move the strip $AB$ vertically from $PQ$ (i.e., $y = c$) to $RS$ (i.e., $y = d$) to cover the entire region $PQRS$. The result of the first integral is integrated w.r.t. $y$ between the limits $c$ and $d$.

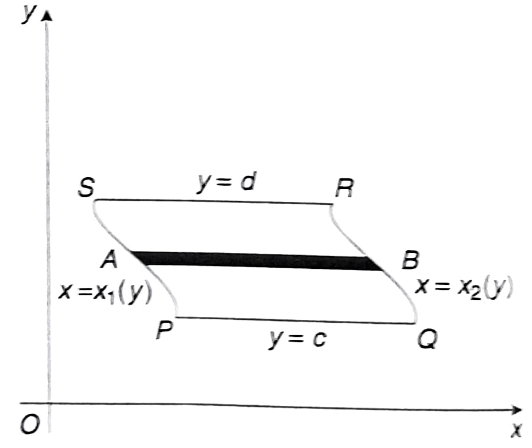


Fig. 5.3

Hence, $\displaystyle\iint_R f(x, y)\,dx\,dy = \int_c^d \left[\int_{x_1(y)}^{x_2(y)} f(x, y)\,dx\right] dy$

***Note:***

(i) If all the four limits are constant then the function $f(x, y)$ can be integrated w.r.t. any variable first. But if $f(x, y)$ is implicit and is discontinuous within or on the boundary of the region of integration then the change of the order of integration will affect the result.

(ii) If all the four limits are constant and $f(x, y)$ is explicit then double integral can be written as product of two single integrals.

(iii) If inner limits depends on $x$ then the function $f(x, y)$ is integrated first w.r.t. $y$ and vice-versa.

## Example 1

*Evaluate* $\int_0^3 \int_0^1 (x^2+3y^2)\,dy\,dx.$

### Solution

$$\begin{aligned}\int_0^3 \int_0^1 (x^2+3y^2)\,dy\,dx &= \int_0^3 \left| x^2 y + y^3 \right|_0^1 dx\\ &= \int_0^3 (x^2+1)\,dx\\ &= \left| \frac{x^3}{3} + x \right|_0^3\\ &= 12.\end{aligned}$$

## Example 2

*Evaluate* $\int_0^1 \int_0^2 (x^2+y^2)\,dy\,dx.$

### Solution

$$\begin{aligned}\int_0^1 \int_0^2 (x^2+y^2)\,dy\,dx &= \int_0^1 \left| x^2 y + \frac{y^3}{3} \right|_0^2 dx\\ &= \int_0^1 \left( 2x^2 + \frac{8}{3} \right) dx\\ &= \left| \frac{2x^3}{3} + \frac{8x}{3} \right|_0^1\\ &= \frac{10}{3}.\end{aligned}$$

## Example 3

*Evaluate* $\int_2^a \int_2^b \frac{dx\,dy}{xy}.$

### Solution

$$\begin{aligned}\int_2^a \int_2^b \frac{dx\,dy}{xy} &= \int_2^a \left( \int_2^b \frac{dx}{x} \right) \frac{dy}{y} = \int_2^a \left| \log x \right|_2^b \frac{1}{y}\,dy\\ &= (\log b - \log 2) \int_2^a \frac{1}{y}\,dy\\ &= \log\left( \frac{b}{2} \right) \cdot \left| \log y \right|_2^a\end{aligned}$$

$$= \log\left(\frac{b}{2}\right)\cdot(\log a - \log 2)$$

$$= \log\left(\frac{b}{2}\right)\cdot\log\left(\frac{a}{2}\right)$$

***Another method:*** Since both the limits are constant and integrand (function) is explicit in $x$ and $y$, the integral can be written as

$$\int_2^a\int_2^b \frac{\mathrm{d}x\,\mathrm{d}y}{xy} = \int_2^a \frac{\mathrm{d}y}{y}\cdot\int_2^b \frac{\mathrm{d}x}{x}$$

$$= |\log y|_2^a \cdot |\log x|_2^b$$

$$= (\log a - \log 2)(\log b - \log 2)$$

$$= \log\left(\frac{a}{2}\right)\cdot\log\left(\frac{b}{2}\right)$$

$$= \log\left(\frac{b}{2}\right)\cdot\log\left(\frac{a}{2}\right).$$

## Example 4

*Evaluate* $\int_0^1\int_1^2 xy\,\mathrm{d}y\,\mathrm{d}x.$

### Solution

$$\int_0^1\int_1^2 xy\,\mathrm{d}y\,\mathrm{d}x = \int_0^1\left\{\int_1^2 y\,\mathrm{d}y\right\}x\,\mathrm{d}x$$

$$= \int_0^1 \left|\frac{y^2}{2}\right|_1^2 x\,\mathrm{d}x$$

$$= \int_0^1\left(\frac{4}{2}-\frac{1}{2}\right)x\,\mathrm{d}x$$

$$= \frac{3}{2}\left|\frac{x^2}{2}\right|_0^1$$

$$= \frac{3}{2}\cdot\frac{1}{2}$$

$$= \frac{3}{4}$$

***Another method:*** Since both the limits are constant and integrand (function) is explicit in $x$ and $y$, the integral can be written as

$$\int_0^1\int_1^2 xy\,\mathrm{d}y\,\mathrm{d}x = \int_0^1 x\,\mathrm{d}x\cdot\int_1^2 y\,\mathrm{d}y$$

$$= \left|\frac{x^2}{2}\right|_0^1\cdot\left|\frac{y^2}{2}\right|_1^2$$

$$= \frac{1}{2}\left(\frac{4}{2} - \frac{1}{2}\right)$$
$$= \frac{3}{4}.$$

## Example 5

*Evaluate* $\int_0^1 \int_0^x \mathrm{d}y\,\mathrm{d}x.$

**Solution**

$$\int_0^1 \int_0^x \mathrm{d}y\,\mathrm{d}x = \int_0^1 \left|y\right|_0^x \mathrm{d}x$$
$$= \int_0^1 x\,\mathrm{d}x$$
$$= \left|\frac{x^2}{2}\right|_0^1$$
$$= \frac{1}{2}$$

## Example 6

*Evaluate* $\int_0^1 \int_0^x e^{\frac{y}{x}}\,\mathrm{d}y\,\mathrm{d}x.$

**Solution**

$$\int_0^1 \int_0^x e^{\frac{y}{x}}\mathrm{d}y\mathrm{d}x = \int_0^1 \left|xe^{\frac{y}{x}}\right|_0^x \mathrm{d}x$$
$$= \int_0^1 x(e-1)\mathrm{d}x$$
$$= \left|\frac{x^2}{2}(e-1)\right|_0^1$$
$$= \frac{1}{2}(e-1)$$

## Example 7

*Evaluate* $\int_0^1 \int_x^{x^2} xy\,\mathrm{d}y\,\mathrm{d}x.$

**Solution**

$$\int_0^1 \int_x^{x^2} xy\,\mathrm{d}y\,\mathrm{d}x = \int_0^1 \left\{\int_x^{x^2} y\,\mathrm{d}y\right\} x\,\mathrm{d}x$$

$$= \int_0^1 \left| \frac{y^2}{2} \right|_x^{x^2} x\,dx$$

$$= \frac{1}{2}\int_0^1 \left[(x^2)^2 - x^2\right] x\,dx$$

$$= \frac{1}{2}\int_0^1 (x^5 - x^3)dx$$

$$= \frac{1}{2}\left| \frac{x^6}{6} - \frac{x^4}{4} \right|_0^1$$

$$= \frac{1}{2}\left(\frac{1}{6} - \frac{1}{4}\right)$$

$$= -\frac{1}{24}.$$

## Example 8

*Evaluate* $\displaystyle\int_0^1 \int_0^{\sqrt{1+x^2}} \frac{dx\,dy}{1+x^2+y^2}$.

### Solution

$$\int_0^1 \left[ \int_0^{\sqrt{1+x^2}} \frac{dy}{\left(\sqrt{1+x^2}\right)^2 + y^2} \right] dx = \int_0^1 \frac{1}{\sqrt{1+x^2}} \left| \tan^{-1} \frac{y}{\sqrt{1+x^2}} \right|_0^{\sqrt{1+x^2}} dx$$

$$= \int_0^1 \frac{1}{\sqrt{1+x^2}} (\tan^{-1} 1 - \tan^{-1} 0) dx$$

$$= \int_0^1 \frac{1}{\sqrt{1+x^2}} \cdot \frac{\pi}{4} dx$$

$\sinh^{-1}$

$$= \frac{\pi}{4} \left| \log\left(x + \sqrt{1+x^2}\right) \right|_0^1$$

$$= \frac{\pi}{4} \log\left(1 + \sqrt{2}\right)$$

## Example 9

*Evaluate* $\displaystyle\int_0^1 \int_0^{\sqrt{\frac{1-y^2}{2}}} \frac{dxdy}{1-x^2-y^2}$.

### Solution

$$\int_0^1 \int_0^{\sqrt{\frac{1-y^2}{2}}} \frac{dx\,dy}{\sqrt{1-x^2-y^2}} = \int_0^1 \left[ \int_0^{\sqrt{\frac{1-y^2}{2}}} \frac{dx}{\sqrt{(1-y^2)-x^2}} \right] dy$$

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$$= \int_0^1 \left| \sin^{-1} \frac{x}{\sqrt{1-y^2}} \right|_0^{\sqrt{\frac{1-y^2}{2}}} dy$$

$$= \int_0^1 \left( \sin^{-1} \frac{1}{\sqrt{2}} - \sin^{-1} 0 \right) dy$$

$$= \frac{\pi}{4} |y|_0^1$$

$$= \frac{\pi}{4}$$

## Example 10

*Show that* $\int_0^1 dx \int_0^1 \frac{x-y}{(x+y)^3} dy \neq \int_0^1 dy \int_0^1 \frac{x-y}{(x+y)^3} dx.$

### *Solution*

Consider,

$$\text{LHS} = \int_0^1 dx \int_0^1 \frac{x-y}{(x+y)^3} dy$$

$$= \int_0^1 dx \int_0^1 \frac{2x-(x+y)}{(x+y)^3} dy$$

$$= \int_0^1 \int_0^1 \left[ \frac{2x}{(x+y)^3} - \frac{1}{(x+y)^2} \right] dy\, dx$$

$$= \int_0^1 \left| 2x \left[ \frac{1}{-2(x+y)^2} \right] + \frac{1}{x+y} \right|_0^1 dx$$

$$= \int_0^1 \left[ -\frac{x}{(x+1)^2} + \frac{1}{x+1} + \frac{1}{x} - \frac{1}{x} \right] dx$$

$$= \int_0^1 \frac{1}{(x+1)^2} dx$$

$$= \left| -\frac{1}{x+1} \right|_0^1$$

$$= \frac{1}{2}$$

$$\text{RHS} = \int_0^1 dy \int_0^1 \frac{x-y}{(x+y)^3} dx$$

$$= \int_0^1 dy \int_0^1 \left[ \frac{(x+y)-2y}{(x+y)^3} \right] dx$$

$$= \int_0^1 \int_0^1 \left[ \frac{1}{(x+y)^2} - \frac{2y}{(x+y)^3} \right] dx\, dy$$

$$= \int_0^1 \left| -\frac{1}{x+y} + \frac{y}{(x+y)^2} \right|_0^1 dy$$

$$= \int_0^1 \left[ -\frac{1}{1+y} + \frac{y}{(1+y)^2} + \frac{1}{y} - \frac{1}{y} \right] dy$$

$$= \int_0^1 -\frac{1}{(1+y)^2}\, dy$$

$$= \left| \frac{1}{1+y} \right|_0^1$$

$$= -\frac{1}{2}$$

Hence, $$\int_0^1 dx \int_0^1 \frac{x-y}{(x+y)^3}\, dy \neq \int_0^1 dy \int_0^1 \frac{x-y}{(x+y)^3}\, dx$$

Since $\dfrac{x-y}{(x+y)^3}$ is discontinuous at (0, 0), a point on the boundary of the region (square), change of order of integration does not give the same result.

## EXERCISE 5.1

Evaluate the following:

1. $\int_1^2 \int_{-\sqrt{2-y}}^{\sqrt{2-y}} 2x^2y^2 dx\, dy$

$\left[\text{Ans.}: \dfrac{856}{945}\right]$

2. $\int_0^1 \int_0^y xy\, e^{x-2} dx\, dy$

$\left[\text{Ans.}: \dfrac{1}{4e}\right]$

3. $\int_0^1 \int_0^x e^{x+y} dx dy$

$\left[\text{Ans.}: \dfrac{1}{2}(e-1)^2\right]$

4. $\int_{10}^1 \int_0^{\frac{1}{y}} ye^{xy} dx\, dy$

[Ans.: $9(1-e)$]

5. $\int_1^{\log 8} \int_0^{\log y} e^{x+y} dx\, dy$

[Ans.: $8(\log 8 - 1)$]

6. $\int_0^1 \int_{y^2}^{y} (1+xy^2) dx\, dy$

$\left[\text{Ans.}: \dfrac{41}{210}\right]$

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7. $\int_0^{2a}\int_0^{\sqrt{2ax-x^2}} xy\,dy\,dx$

$$\left[\text{Ans.}: \frac{2a^4}{3}\right]$$

## Working Rule for Evaluation of Double Integrals

1. If the region is bounded by more than one curve then find the points of intersection of all the curves.
2. Draw all the curves and mark their point of intersection.
3. Identify the region bounded by all the curves.
4. Draw a vertical or horizontal strip in the region whichever makes the integration easier.
5. The vertical strip starts from the lowest part of the region and terminates on the highest part of the region.
6. For vertical strip: (i) The lower limit of $y$ is obtained from the curve where the vertical strip starts and the upper limit of $y$ is obtained from the curve where it terminates.
   (ii) The lower limit of $x$ is obtained from the leftmost point of the region and the upper limit of $x$ is obtained from the rightmost point of the region.
7. The horizontal strip starts from the left part of the region and terminates on the right part of the region.
8. For horizontal strip: (i) The lower limit of $x$ is obtained from the curve where the horizontal strip starts and upper limit is obtained from the curve where it terminates.
   (ii) The lower limit of $y$ is obtained from the lowest point of the region and the upper limit of $y$ is obtained from the highest point of the region.
9. If variation along the strip changes within the region then the region is divided into parts.

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## Example 1

*Evaluate* $\iint e^{ax+by}\,dx\,dy$, *over the triangle bounded by* $x = 0,\ y = 0,\ ax + by = 1$.

### Solution

1. The region of integration is the $\Delta OPQ$.
2. The integration can be done w.r.t. any variable first. Draw a vertical strip $AB$ parallel to $y$-axis which starts from $x$-axis and terminates on the line $ax + by = 1$.
3. Limits of $y$: $y = 0$ to $y = \dfrac{1-ax}{b}$

   Limits of $x$: $x = 0$ to $x = \dfrac{1}{a}$

$$I = \int_0^{\frac{1}{a}} \int_0^{\frac{1-ax}{b}} e^{ax+by}\, dx\, dy$$

$$= \int_0^{\frac{1}{a}} e^{ax} \int_0^{\frac{1-ax}{b}} e^{by}\, dy\, dx$$

$$= \int_0^{\frac{1}{a}} e^{ax} \left| \frac{e^{by}}{b} \right|_0^{\frac{1-ax}{b}}$$

$$= \frac{1}{b} \int_0^{\frac{1}{a}} e^{ax} \left[ e^{(1-ax)} - 1 \right] dx \quad = e^{1-ax+ax} - e^{ay}$$

$$= \frac{1}{b} \int_0^{\frac{1}{a}} (e - e^{ax}) dx$$

$$= \frac{1}{b} \left| ex - \frac{e^{ax}}{a} \right|_0^{\frac{1}{a}}$$

$$= \frac{1}{b} \left( \frac{e}{a} - \frac{e}{a} + \frac{1}{a} \right)$$

$$= \frac{1}{ab}.$$

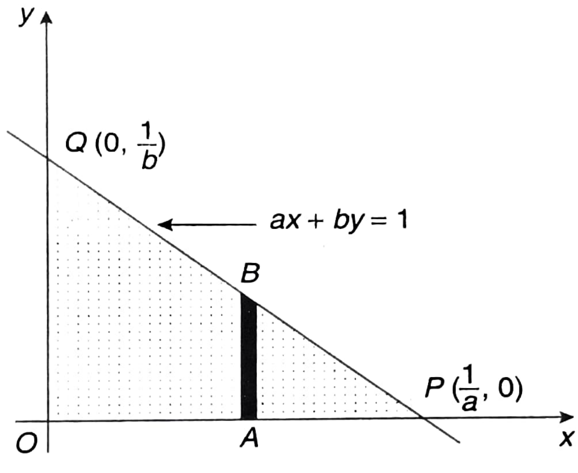


**Fig. 5.4**

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## Example 2

*Evaluate* $\iint \frac{xy}{\sqrt{1-y^2}}\, dx\, dy$ *over the first quadrant of the circle* $x^2 + y^2 = 1$.

### Solution

1. The region of integration is *OPQ*.
2. The integration can be done w.r.t any variable first. Draw a vertical strip *AB* parallel to *y*-axis which starts from *x*-axis and terminates on the circle $x^2 + y^2 = 1$.
3. Limits of $y$: $y = 0$ to $y = \sqrt{1-x^2}$

   Limits of $x$: $x = 0$ to $x = 1$

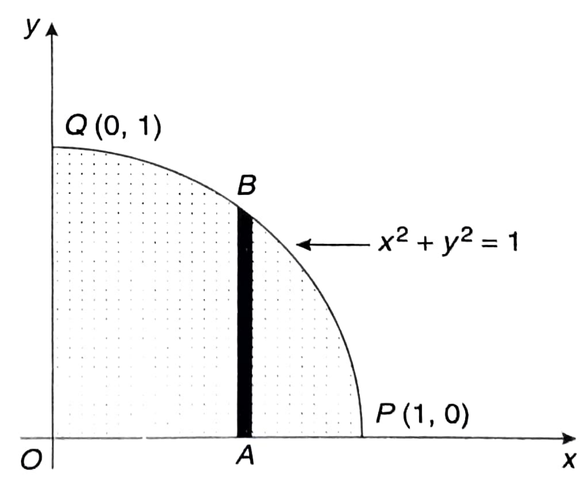


**Fig. 5.5**

$$I = \int_0^1 \int_0^{\sqrt{1-x^2}} \frac{xy}{\sqrt{1-y^2}}\, dx\, dy$$

$$= \int_0^1 x \int_0^{\sqrt{1-x^2}} -\frac{1}{2}(1-y^2)^{-\frac{1}{2}}(-2y)\, dy\, dx$$

$$= -\frac{1}{2} \int_0^1 x \left| 2(1-y^2)^{\frac{1}{2}} \right|_0^{\sqrt{1-x^2}} dx \qquad \left[ \because \int [f(x)]^n f'(x) dx = \frac{[f(x)]^{n+1}}{n+1} \right]$$

$$= -\frac{1}{2} \int_0^1 2x(x-1) dx$$

$$= -\left|\frac{x^3}{3} - \frac{x^2}{2}\right|_0^1$$

$$= -\left(\frac{1}{3} - \frac{1}{2}\right)$$

$$= \frac{1}{6}$$

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## Example 3

*Evaluate* $\iint (a-x)^2 \, dx \, dy$, *over the right half of the circle* $x^2 + y^2 = a^2$.

### Solution

1. The region of integration is *PQR*.
2. The integration can be done w.r.t. any variable first. Draw a vertical strip *AB* parallel to $y$-axis which starts from the part of the circle $x^2 + y^2 = a^2$ below $x$-axis and terminates on the part of the circle $x^2 + y^2 = a^2$ above $x$-axis.
3. Limits of

$y : y = -\sqrt{a^2 - x^2}$ to $y = \sqrt{a^2 - x^2}$

Limits of $x : x = 0$ to $x = a$

$$I = \int_0^a \int_{-\sqrt{a^2-x^2}}^{\sqrt{a^2-x^2}} (a-x)^2 \, dx \, dy$$

$$= \int_0^a (a-x)^2 \int_{-\sqrt{a^2-x^2}}^{\sqrt{a^2-x^2}} dy \, dx$$

$$= \int_0^a (a-x)^2 \left|y\right|_{-\sqrt{a^2-x^2}}^{\sqrt{a^2-x^2}} dx$$

$$= \int_0^a (a^2 + x^2 - 2ax) \cdot 2\sqrt{a^2 - x^2} \, dx$$

$$= 2\int_0^a (a^2 + x^2 - 2ax)\sqrt{a^2 - x^2} \, dx$$

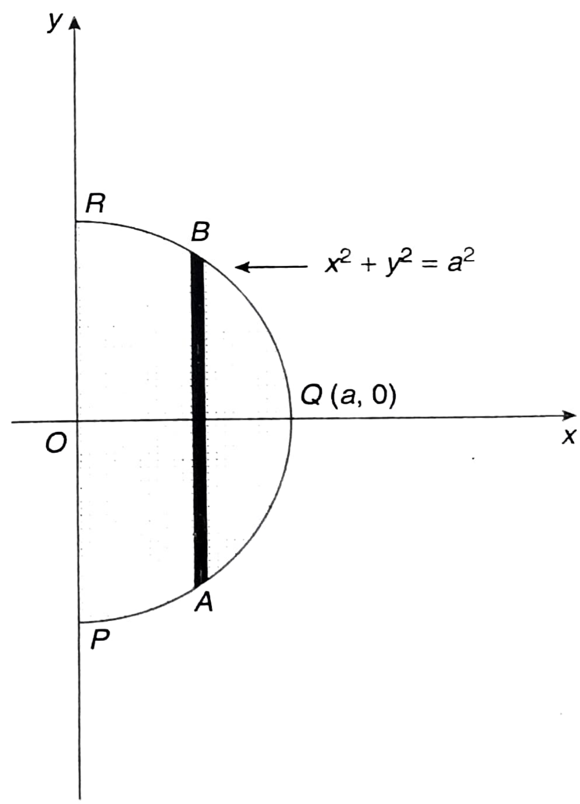


Fig. 5.6

Putting $x = a\sin\theta$, $dx = a\cos\theta \, d\theta$

When $x = 0$, $\theta = 0$

When $x = a$, $\theta = \frac{\pi}{2}$

$$I = 2\int_0^{\frac{\pi}{2}} (a^2 + a^2\sin^2\theta - 2a^2\sin\theta) \cdot a\cos\theta \cdot a\cos\theta \, d\theta$$

$$= 2a^4 \int_0^{\frac{\pi}{2}} (\cos^2\theta + \sin^2\theta\cos^2\theta - 2\sin\theta\cos^2\theta) d\theta$$

$$= 2a^4 \left[\int_0^{\frac{\pi}{2}} \cos^2\theta \, d\theta + \int_0^{\frac{\pi}{2}} \sin^2\theta\cos^2\theta \, d\theta + 2\int_0^{\frac{\pi}{2}} \cos^2\theta(-\sin\theta) d\theta\right]$$

$$= 2a^4\left[\left(\frac{1}{2}\cdot\frac{\pi}{2}\right)+\left(\frac{1}{4}\cdot\frac{1}{2}\cdot\frac{\pi}{2}\right)+2\left|\frac{\cos^3\theta}{3}\right|_0^{\frac{\pi}{2}}\right]$$

$$\left[\text{Using reduction formula and } \int[f(\theta)]^n f'(\theta)\mathrm{d}\theta = \frac{[f(\theta)]^{n+1}}{n+1}\right]$$

$$= 2a^4\left[\frac{\pi}{4}+\frac{\pi}{16}+\frac{2}{3}\left(\cos^3\frac{\pi}{2}-\cos 0\right)\right]$$

$$= a^4\left[\frac{\pi}{2}+\frac{\pi}{8}-\frac{4}{3}\right]$$

$$= a^4\left[\frac{5\pi}{8}-\frac{4}{3}\right].$$

## Example 4

*Evaluate* $\iint xy\left(\frac{x^2}{a^2}+\frac{y^2}{b^2}\right)^{\frac{n}{2}} \mathrm{d}x\,\mathrm{d}y$, *over the first quadrant of the ellipse* $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1.$

### Solution

1. The region of integration is $OPQ$.
2. The integration can be done w.r.t any variable first. Draw a vertical strip $AB$ parallel to $y$-axis which starts from $x$-axis and terminates on the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1.$
3. Limits of $y$: $y = 0$ to $y = b\sqrt{1-\frac{x^2}{a^2}}$

   Limits of $x$: $x = 0$ to $x = a$

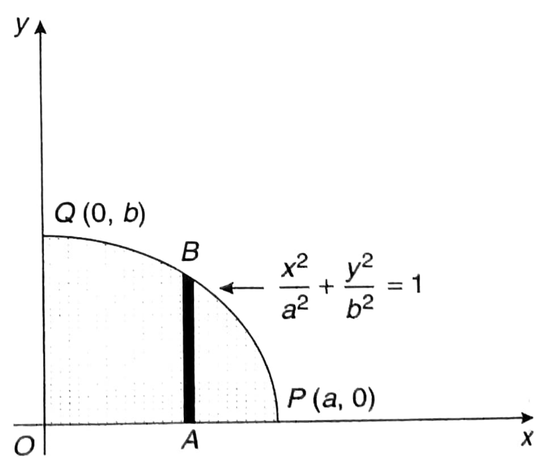


**Fig. 5.7**

$$I = \int_0^a\int_0^{b\sqrt{1-\frac{x^2}{a^2}}} xy\left(\frac{x^2}{a^2}+\frac{y^2}{b^2}\right)^{\frac{n}{2}} \mathrm{d}x\,\mathrm{d}y$$

$$= \int_0^a x\int_0^{b\sqrt{1-\frac{x^2}{a^2}}} \frac{b^2}{2}\left(\frac{x^2}{a^2}+\frac{y^2}{b^2}\right)^{\frac{n}{2}} \frac{2y}{b^2}\,\mathrm{d}y\,\mathrm{d}x$$

$$= \frac{b^2}{2}\int_0^a x\left|\frac{1}{\left(\frac{n}{2}+1\right)}\left(\frac{x^2}{a^2}+\frac{y^2}{b^2}\right)^{\frac{n}{2}+1}\right|_0^{b\sqrt{1-\frac{x^2}{a^2}}} \mathrm{d}x \qquad \left[\because \int[f(y)]^n f'(y)\,\mathrm{d}y = \frac{[f(y)]^{n+1}}{n+1}\right]$$

$$= \frac{b^2}{n+2}\left|\frac{x^2}{2} - \frac{1}{a^{n+2}} \cdot \frac{x^{n+4}}{n+4}\right|_0^a$$

$$= \frac{b^2}{n+2}\left|\frac{a^2}{2} - \frac{1}{a^{n+2}} \cdot \frac{a^{n+4}}{n+4}\right|$$

$$= \frac{a^2 b^2}{(n+2)} \cdot \frac{(n+2)}{2(n+4)}$$

$$= \frac{a^2 b^2}{2(n+4)}$$

## Example 5

*Evaluate* $\iint (x^2 + y^2)\,dx\,dy$ *over the ellipse* $2x^2 + y^2 = 1$.

### Solution

1. The region of integration is *PQRS*, the ellipse $2x^2 + y^2 = 1$ or $\frac{x^2}{\left(\frac{1}{\sqrt{2}}\right)^2} + \frac{y^2}{1^2} = 1$ with $\frac{1}{\sqrt{2}}$ and 1 as its axes.
2. The integration can be done w.r.t any variable first. Draw a vertical strip *AB* parallel to $y$-axis which starts from the part of the ellipse $2x^2 + y^2 = 1$ below $x$-axis and terminates on the part of the ellipse $2x^2 + y^2 = 1$ above $x$-axis.
3. Limits of

   $y : y = -\sqrt{1-2x^2}$ to $y = \sqrt{1-2x^2}$

   Limits of $x : x = -\frac{1}{\sqrt{2}}$ to $x = \frac{1}{\sqrt{2}}$

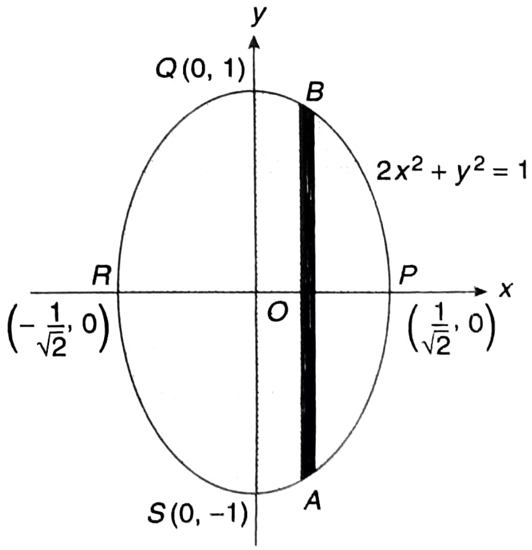


Fig. 5.8

$$I = \int_{-\frac{1}{\sqrt{2}}}^{\frac{1}{\sqrt{2}}} \int_{-\sqrt{1-2x^2}}^{\sqrt{1-2x^2}} (x^2 + y^2)\,dy\,dx$$

$$= \int_{-\frac{1}{\sqrt{2}}}^{\frac{1}{\sqrt{2}}} \left|x^2 y + \frac{y^3}{3}\right|_{-\sqrt{1-2x^2}}^{\sqrt{1-2x^2}} dx$$

$$= \int_{-\frac{1}{\sqrt{2}}}^{\frac{1}{\sqrt{2}}} 2\left[x^2\sqrt{1-2x^2} + \frac{1}{3}(1-2x^2)^{\frac{3}{2}}\right] dx$$

$$= 4\int_0^{\frac{1}{\sqrt{2}}} \left[x^2\sqrt{1-2x^2} + \frac{1}{3}\left(1-2x^2\right)^{\frac{3}{2}}\right] dx$$

Putting $2x^2 = \cos^2\theta,\ x = \frac{1}{\sqrt{2}}\cos\theta,\ dx = -\frac{1}{\sqrt{2}}\sin\theta\, d\theta$

When $x = 0,\ \cos\theta = 0,\ \theta = \frac{\pi}{2}$

When $x = \frac{1}{\sqrt{2}},\ \cos\theta = 1,\ \theta = 0$

$$4\int_0^{\frac{1}{\sqrt{2}}}\left[x^2\sqrt{1-2x^2} + \frac{1}{3}(1-2x^2)^{\frac{3}{2}}\right]dx$$

$$= 4\int_{\frac{\pi}{2}}^{0}\left[\frac{\cos^2\theta}{2}\sqrt{1-\cos^2\theta} + \frac{1}{3}(1-\cos^2\theta)^{\frac{3}{2}}\right]\left(-\frac{1}{\sqrt{2}}\sin\theta\, d\theta\right)$$

$$= 2\sqrt{2}\left[\frac{1}{2}\int_0^{\frac{\pi}{2}}\cos^2\theta\sin^2\theta\, d\theta + \frac{1}{3}\int_0^{\frac{\pi}{2}}\sin^4\theta\, d\theta\right]$$

$$= 2\sqrt{2}\left[\frac{1}{2}\left(\frac{1}{4}\cdot\frac{1}{2}\cdot\frac{\pi}{2}\right) + \frac{1}{3}\left(\frac{3}{4}\cdot\frac{1}{2}\cdot\frac{\pi}{2}\right)\right] \quad \text{[Using reduction formula]}$$

$$= \frac{3\sqrt{2}\pi}{16}.$$

## Example 6

*Evaluate* $\iint (x^2 - y^2)\,dx\,dy$ *over the triangle with the vertices* (0, 1), (1, 1), (1, 2).

### Solution

1. The region of integration is $\Delta PQR$.
2. Equation of the line $PQ$ is $y = 1$.
   Equation of the line $PR$ is
   $$y - 1 = \frac{2-1}{1-0}(x-0) = x$$
   $$y = x + 1$$
3. The integration can be done w.r.t. any variable first. Draw a vertical strip $AB$ parallel to $y$-axis which starts from the line $y = 1$ and terminates on the line $y = x + 1$.
4. Limits of $y$: $y = 1$ to $y = x + 1$
   Limits of $x$: $x = 0$ to $x = 1$

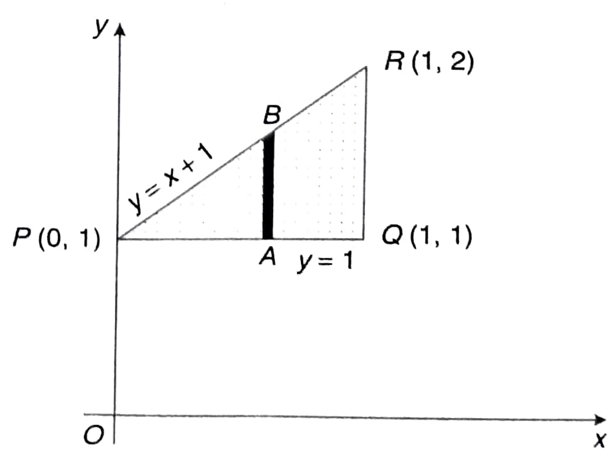


Fig. 5.9

$$I = \int_0^1 \int_1^{x+1} (x^2 - y^2)\,dy\,dx$$

$$= \int_0^1 \left| x^2 y - \frac{y^3}{3} \right|_1^{x+1} dx$$

$$= \int_0^1 \left[ x^2(x+1) - \frac{(x+1)^3}{3} - x^2 + \frac{1}{3} \right] dx$$

$$= \left| \frac{x^4}{4} + \frac{x^3}{3} - \frac{(x+1)^4}{12} - \frac{x^3}{3} + \frac{x}{3} \right|_0^1$$

$$= \frac{1}{4} + \frac{1}{3} - \frac{16}{12} + \frac{1}{12}$$

$$= -\frac{2}{3}.$$

## Example 7

*Evaluate* $\iint e^{y^2}\,dx\,dy$ *over the region bounded by the triangle with vertices* (0, 0), (2, 1), (0, 1).

### Solution

1. The region of integration is $\Delta OPQ$.
2. Equation of the line $OQ$ is $y = \frac{x}{2}$ or $x = 2y$.
3. Here, it is easier to integrate w.r.t. $x$ first than $y$. Draw a horizontal strip $AB$ parallel to $x$-axis which starts from $y$-axis and terminates on the line $x = 2y$.
4. Limits of $x$: $x = 0$ to $x = 2y$
   Limits of $y$: $y = 0$ to $y = 1$

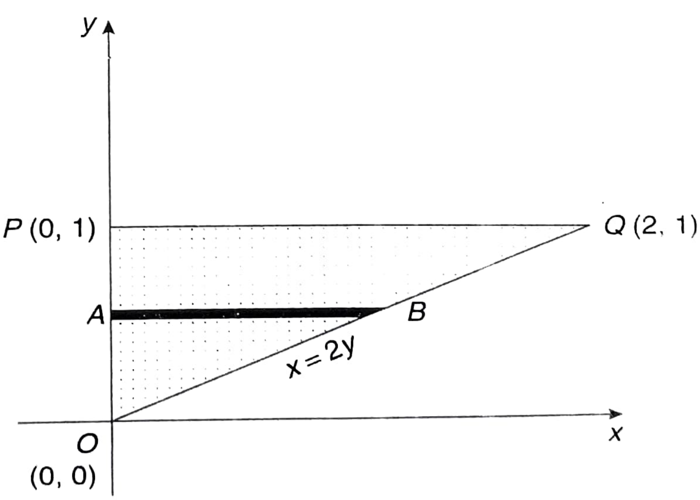


Fig. 5.10

$$I = \int_0^1 \int_0^{2y} e^{y^2}\,dx\,dy$$

$$= \int_0^1 e^{y^2} \int_0^{2y} dx\,dy$$

$$= \int_0^1 e^{y^2} \left| x \right|_0^{2y} dy$$

$$= \int_0^1 e^{y^2} \cdot 2y\,dy$$

$$= \left| e^{y^2} \right|_0^1 \qquad \left[ \because \int e^{f(y)} f'(y)\,dy = e^{f(y)} \right]$$

$$= e - 1$$

## Example 8

*Evaluate* $\iint \frac{2xy^5}{\sqrt{1+x^2y^2-y^4}}\mathrm{d}x\,\mathrm{d}y$ *over the triangle having vertices* (0, 0), (1, 1) *and* (0, 1).

### Solution

1. The region of integration is the $\Delta OPQ$.
2. Equation of the line $OP$ is $y = x$.
3. Here, it is easier to integrate w.r.t. $x$ first than with $y$. Draw a horizontal strip $AB$ parallel to $x$-axis which starts from $y$-axis and terminates on the line $y = x$.
4. Limits of $x : x = 0$ to $x = y$
   Limits of $y : y = 0$ to $y = 1$

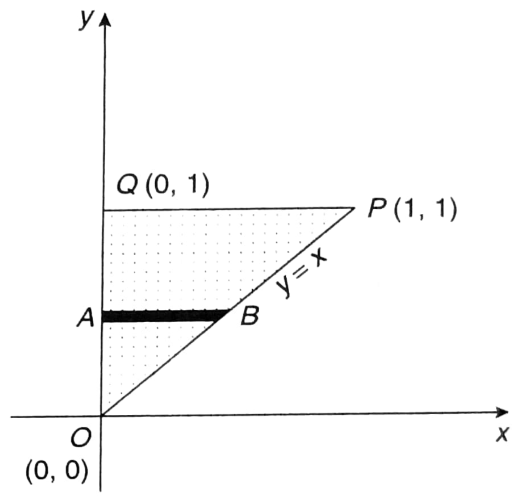


Fig. 5.11

$$I = \int_0^1\int_0^y \frac{2xy^5}{\sqrt{1+x^2y^2-y^4}}\mathrm{d}x\,\mathrm{d}y$$

$$= \int_0^1 y^3\int_0^y (1+x^2y^2-y^4)^{-\frac{1}{2}}\cdot 2xy^2\mathrm{d}x\,\mathrm{d}y$$

$$= \int_0^1 y^3\left|2(1+x^2y^2-y^4)^{\frac{1}{2}}\right|_0^y \mathrm{d}y \qquad \left[\because \int [f(x)]^n f'(x)\mathrm{d}x = \frac{[f(x)]^{n+1}}{n+1}\right]$$

$$= \int_0^1 y^3\cdot 2\left[1-(1-y^4)^{\frac{1}{2}}\right]\mathrm{d}y$$

$$= \int_0^1 2y^3\mathrm{d}y - 2\int_0^1 (1-y^4)^{\frac{1}{2}}y^3\mathrm{d}y$$

$$= 2\left|\frac{y^4}{4}\right|_0^1 - 2\int_0^1 (1-y^4)^{\frac{1}{2}}\frac{(-4y^3)}{-4}\mathrm{d}y$$

$$= \frac{1}{2}+\frac{1}{2}\left|\frac{2}{3}(1-y^4)^{\frac{3}{2}}\right|_0^1 \qquad \left[\because \int [f(x)]^n f'(x)\mathrm{d}x = \frac{[f(x)]^{n+1}}{n+1}\right]$$

$$= \frac{1}{2}-\frac{1}{3}$$

$$= \frac{1}{6}.$$

## Example 9

*Evaluate* $\iint (x^2+y^2)\mathrm{d}x\,\mathrm{d}y$, *over the region bounded by the lines* $y = 4x,\ x + y = 3,\ y = 0,\ y = 2.$

9



### Solution

1. The region of integration is *OPQR*.
2. The integration can be done w.r.t. any variable first. But in case of vertical strip we need to divide the region into three parts. Therefore, draw a horizontal strip *AB* parallel to $x$-axis which starts from the line $y = 4x$ and terminates on the line $x + y = 3$.

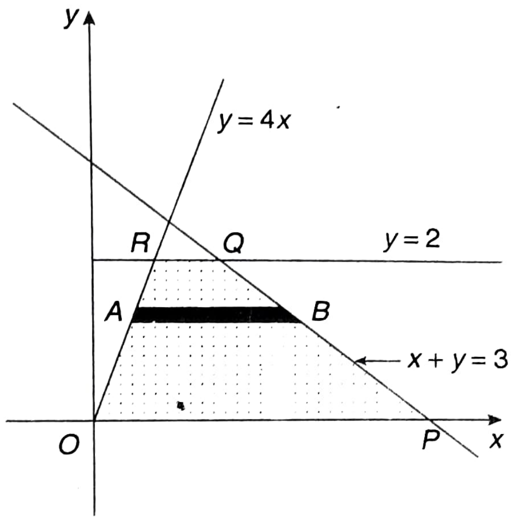


**Fig. 5.12**

3. Limits of $x : x = \frac{y}{4}$ to $x = 3 - y$

   Limits of $y$: $y = 0$ to $y = 2$

$$I = \int_0^2 \int_{\frac{y}{4}}^{3-y} (x^2 + y^2)\,dx\,dy$$

$$= \int_0^2 \left| \frac{x^3}{3} + xy^2 \right|_{\frac{y}{4}}^{3-y} dy$$

$$= \int_0^2 \left[ \frac{(3-y)^3}{3} + (3-y)y^2 - \frac{1}{3}\cdot\frac{y^3}{64} - \frac{y^3}{4} \right] dy$$

$$= \int_0^2 \left[ \frac{(3-y)^3}{3} + 3y^2 - \frac{241}{192} y^3 \right] dy$$

$$= \left| \frac{1}{3}\cdot\frac{(3-y)^4}{-4} + 3\cdot\frac{y^3}{3} - \frac{241}{192}\cdot\frac{y^4}{4} \right|_0^2$$

$$= -\frac{1}{12} + 8 - \frac{241}{192}\cdot 4 - \left(-\frac{27}{4}\right)$$

$$= \frac{463}{48}.$$

## Example 10

*Evaluate* $\iint \frac{y\,dx\,dy}{(a-x)\sqrt{ax - y^2}}$ *over the region bounded by the parabola* $y^2 = x$ *and the line* $y = x$.

### Solution

1. The region of integration is *OPQ*.
2. The points of intersection of $y^2 = x$ and $y = x$ are obtained as

$$x^2 = x$$
$$x(x-1) = 0$$
$$x = 0, 1$$
$$\therefore y = 0, 1$$

The points of intersection are $O\ (0, 0)$ and $P\ (1, 1)$.

3. Here, it is easier to integrate w.r.t. $y$ first than $x$. Draw a vertical strip $AB$ parallel to $y$-axis, which starts from the line $y = x$ and terminates on the parabola $y^2 = x$.

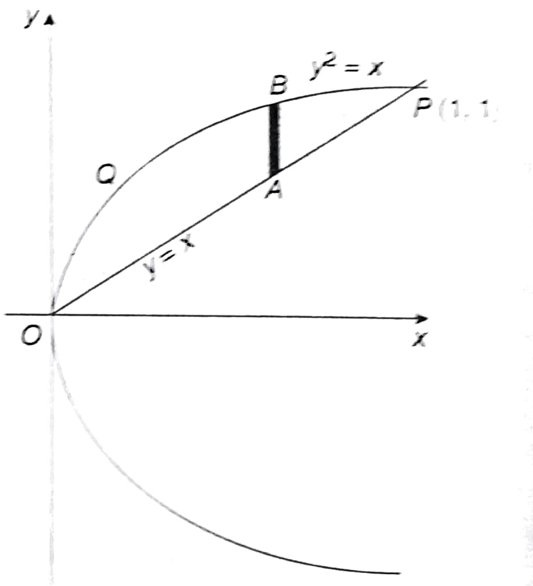


Fig. 5.13

4. Limits of $y$: $y = x$ to $y = \sqrt{x}$

Limits of $x$: $x = 0$ to $x = 1$

$$I = \int_0^1 \int_x^{\sqrt{x}} \frac{y\,dx\,dy}{(a-x)\sqrt{ax-y^2}}$$

$$= \int_0^1 \frac{1}{(a-x)} \int_x^{\sqrt{x}} \left(-\frac{1}{2}\right)(ax-y^2)^{-\frac{1}{2}}(-2y)dy\,dx$$

$$= -\frac{1}{2}\int_0^1 \frac{1}{(a-x)} \left| 2(ax-y^2)^{\frac{1}{2}} \right|_x^{\sqrt{x}} dx \qquad \left[\because \int [f(y)]^n f'(y)dy = \frac{[f(y)]^{n+1}}{n+1}\right]$$

$$= -\int_0^1 \frac{1}{(a-x)}\left[(ax-x)^{\frac{1}{2}} - (ax-x^2)^{\frac{1}{2}}\right]dx$$

$$= -\int_0^1 \frac{\sqrt{x}}{a-x}\left(\sqrt{a-1} - \sqrt{a-x}\right)dx$$

Putting $x = a\sin^2\theta$, $dx = 2a\sin\theta\cos\theta\,d\theta$

When $x = 0$, $\theta = 0$

When $x = 1$, $\theta = \sin^{-1}\dfrac{1}{\sqrt{a}}$

$$I = -\int_0^{\sin^{-1}\frac{1}{\sqrt{a}}} \frac{\sqrt{a}\sin\theta}{a\cos^2\theta}\left(\sqrt{a-1} - \sqrt{a}\cos\theta\right)2a\sin\theta\cos\theta\,d\theta$$

$$= -2\sqrt{a}\int_0^{\sin^{-1}\frac{1}{\sqrt{a}}} \frac{\sin^2\theta}{\cos\theta}\left(\sqrt{a-1} - \sqrt{a}\cos\theta\right)d\theta$$

$$= -2\sqrt{a}\int_0^{\sin^{-1}\frac{1}{\sqrt{a}}} \left[\left(\frac{1-\cos^2\theta}{\cos\theta}\right)\sqrt{a-1} - \sqrt{a}\sin^2\theta\right]d\theta$$

$$= -2\sqrt{a}\int_0^{\sin^{-1}\frac{1}{\sqrt{a}}} \left[\sqrt{a-1}(\sec\theta - \cos\theta) - \frac{\sqrt{a}}{2}(1-\cos 2\theta)\right]d\theta$$

$$= -2\sqrt{a}\left|\sqrt{a-1}[\log(\sec\theta + \tan\theta) - \sin\theta] - \frac{\sqrt{a}\theta}{2} + \frac{\sqrt{a}\sin 2\theta}{4}\right|_0^{\sin^{-1}\frac{1}{\sqrt{a}}}$$

$$= -2\sqrt{a}\left|\sqrt{a-1}\left[\log\left(\frac{1+\sin\theta}{\cos\theta}\right)-\sin\theta\right]-\frac{\sqrt{a}\theta}{2}+\frac{\sqrt{a}\sin\theta\cos\theta}{2}\right|_{0}^{\sin^{-1}\frac{1}{\sqrt{a}}}$$

$$= -2\sqrt{a}\left[\sqrt{a-1}\left(\log\frac{1+\frac{1}{\sqrt{a}}}{\sqrt{1-\frac{1}{a}}}-\frac{1}{\sqrt{a}}\right)-\frac{\sqrt{a}\sin^{-1}\frac{1}{\sqrt{a}}}{2}+\frac{\sqrt{a}}{2}\cdot\frac{1}{\sqrt{a}}\cdot\sqrt{1-\frac{1}{a}}\right]$$

$$= -2\sqrt{a(a-1)}\log\frac{\sqrt{a}+1}{\sqrt{a-1}}+\sqrt{a-1}+a\sin^{-1}\frac{1}{\sqrt{a}}.$$

## Example 11

*Evaluate* $\iint y\,dx\,dy$ *over the region enclosed by the parabola* $x^2 = y$ *and the line* $y = x + 2$.

### Solution

1. The region of integration is $POQ$.
2. The points of intersection of $x^2 = y$ and $y = x + 2$ are obtained as

$$x^2 = x+2$$
$$x^2 - x - 2 = 0$$
$$(x-2)(x+1) = 0$$
$$x = 2, -1$$
$$\therefore\ y = 4, 1$$

The points of intersection are $P(-1, 1)$ and $Q(2, 4)$.

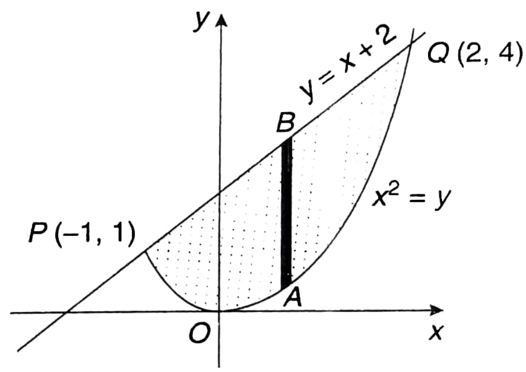


**Fig. 5.14**

3. The integration can be done w.r.t. any variable first. Draw a vertical strip $AB$ parallel to $y$-axis which starts from the parabola $x^2 = y$ and terminates on the line $y = x + 2$.
4. Limits of $y$ : $y = x^2$ to $y = x + 2$
   Limits of $x$ : $x = -1$ to $x = 2$

$$I = \int_{-1}^{2}\int_{x^2}^{x+2} y\,dy\,dx$$

$$= \int_{-1}^{2}\left|\frac{y^2}{2}\right|_{x^2}^{x+2} dx$$

$$= \frac{1}{2}\int_{-1}^{2}\left[(x+2)^2 - x^4\right]dx$$

$$= \frac{1}{2}\left|\frac{(x+2)^3}{3}-\frac{x^5}{5}\right|_{-1}^{2}$$

$$= \frac{1}{2}\left(\frac{64}{3}-\frac{32}{5}-\frac{1}{3}-\frac{1}{5}\right)$$

$$= \frac{36}{5}.$$

## Example 12

*Evaluate* $\iint xy(x+y)\mathrm{d}x\,\mathrm{d}y$, *over the region enclosed by the parabolas* $x^2 = y,\ y^2 = -x$.

### Solution

1. The region of integration is $OPQ$.
2. The points of intersection of the parabola $x^2 = y$, and $y^2 = -x$ are obtained as
$$y^4 = y$$
$$y = 0, 1$$
$$\therefore\ x = 0, -1.$$
The points of intersection are $O$ (0, 0) and $Q$ (−1, 1).
3. Here, it is easier to integrate w.r.t. $x$ first. Draw a horizontal strip $AB$ parallel to $x$-axis, which starts from the parabola $x^2 = y$ and terminates on the parabola $y^2 = -x$.
4. Limits of $x$ : $x = -\sqrt{y}$ to $x = -y^2$

   Limits of $y$ : $y = 0$ to $y = 1$

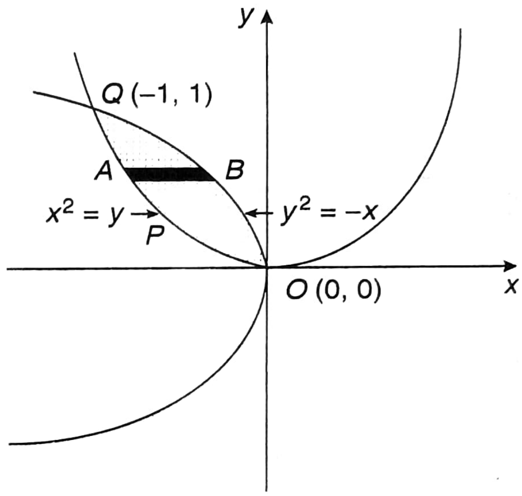


Fig. 5.15

$$I = \int_0^1 y \int_{-\sqrt{y}}^{-y^2} xy(x+y)\mathrm{d}x\,\mathrm{d}y$$
$$= \int_0^1 y \int_{-\sqrt{y}}^{-y^2} (x^2+xy)\mathrm{d}x\,\mathrm{d}y$$
$$= \int_0^1 y \left|\frac{x^3}{3} + \frac{x^2 y}{2}\right|_{-\sqrt{y}}^{-y^2} \mathrm{d}y$$
$$= \int_0^1 \left(\frac{-y^7}{3} + \frac{y^6}{2} + \frac{y^{\frac{5}{2}}}{3} - \frac{y^3}{2}\right)\mathrm{d}y$$
$$= \left|-\frac{y^8}{24} + \frac{y^7}{14} + \frac{2y^{\frac{7}{2}}}{21} - \frac{y^4}{8}\right|_0^1$$
$$= 0$$

## Example 13

*Evaluate* $\iint xy\,\mathrm{d}x\,\mathrm{d}y$ *over the region enclosed by the x-axis, the line* $x = 2a$ *and the parabola* $x^2 = 4ay$.

### Solution

1. The region of integration is $OPQ$.

2. The point of intersection of the parabola $x^2 = 4ay$ and the line $x = 2a$ is obtained as

$$4a^2 = 4ay$$
$$y = a$$

The point of intersection is $Q\ (2a, a)$.

3. The integration can be done w.r.t. any variable first. Draw a vertical strip $AB$ parallel to $y$-axis, which starts from $x$-axis and terminates on the parabola $x^2 = 4ay$.

4. Limits of $y : y = 0$ to $y = \frac{x^2}{4a}$

Limits of $x : x = 0$ to $x = 2a$

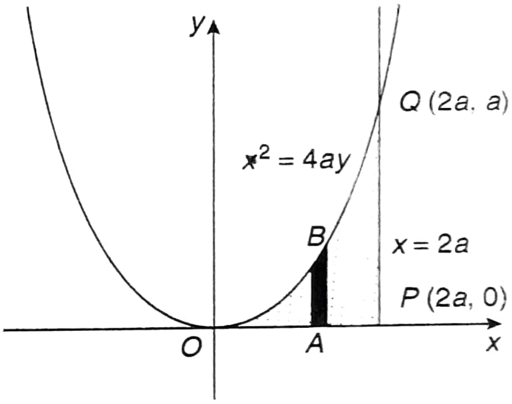


**Fig. 5.16**

$$I = \int_0^{2a}\int_0^{\frac{x^2}{4a}} xy\,dx\,dy$$
$$= \int_0^{2a} x\int_0^{\frac{x^2}{4a}} y\,dy\,dx$$
$$= \int_0^{2a} x\left|\frac{y^2}{2}\right|_0^{\frac{x^2}{4a}} dx$$
$$= \int_0^{2a} x\cdot\frac{x^4}{32a^2}\,dx$$
$$= \frac{1}{32a^2}\left|\frac{x^6}{6}\right|_0^{2a}$$
$$= \frac{1}{32a^2}\cdot\frac{64a^6}{6}$$
$$= \frac{a^4}{3}.$$

## Example 14

*Evaluate* $\iint xy\,dx\,dy$, *over the region enclosed by the circle* $x^2 + y^2 - 2x = 0$, *the parabola* $y^2 = 2x$ *and the line* $y = x$.

### Solution

1. The region of integration is $OPQRO$.
2. (i) The points of intersection of the circle $x^2 + y^2 - 2x = 0$ and the line $y = x$ are obtained as

$$x^2 + x^2 - 2x = 0$$
$$x = 0, 1$$
$$\therefore y = 0, 1$$

The points of intersection are $O\ (0, 0)$ and $P\ (1, 1)$.

(ii) The point of intersection of the circle $x^2 + y^2 - 2x = 0$ and the parabola $y^2 = 2x$ is obtained as

$$x^2 + 2x - 2x = 0$$
$$x = 0$$
$$\therefore\ y = 0$$

The point of intersection is $O$ (0, 0).

(iii) The points of intersection of the parabola $y^2 = 2x$ and the line $y = x$ are obtained as $x^2 = 2x$

$$x = 0, 2$$
$$\therefore\ y = 0, 2$$

The points of intersection are $O$ (0, 0) and $Q$ (2, 2).

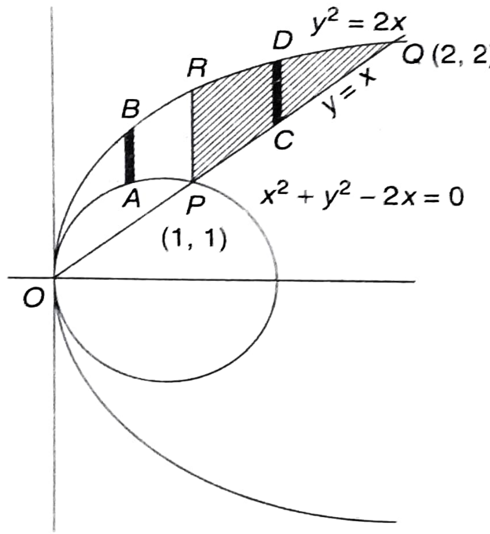


**Fig. 5.17**

3. The integration can be done w.r.t. any variable first. To integrate w.r.t. $y$ first we need to draw a vertical strip in the region. But one vertical strip does not cover the entire region, therefore, divide the region $OPQRO$ into two subregions $OPR$ and $RPQ$ and draw one vertical strip in each subregion.
4. In the subregion $OPR$, strip starts from the circle $x^2 + y^2 - 2x = 0$ and terminates on the parabola $y^2 = 2x$.

   Limits of $y$: $y = \sqrt{2x - x^2}$ to $y = \sqrt{2x}$
   Limits of $x$: $x = 0$ to $x = 1$.
5. In the subregion $RPQ$, strip starts from the line $y = x$ and terminates on the parabola $y^2 = 2x$.

   Limits of $y$: $y = x$ to $y = \sqrt{2x}$
   Limits of $x$: $x = 1$ to $x = 2$

$$
\begin{aligned}
I &= \iint xy\, dx\, dy \\
&= \iint_{OPR} xy\, dx\, dy + \iint_{RPQ} xy\, dx\, dy \\
&= \int_0^1 \int_{\sqrt{2x-x^2}}^{\sqrt{2x}} xy\, dy\, dx + \int_1^2 \int_x^{\sqrt{2x}} xy\, dy\, dx \\
&= \int_0^1 x \int_{\sqrt{2x-x^2}}^{\sqrt{2x}} y\, dy\, dx + \int_1^2 x \int_x^{\sqrt{2x}} y\, dy\, dx \\
&= \int_0^1 x \left| \frac{y^2}{2} \right|_{\sqrt{2x-x^2}}^{\sqrt{2x}} dx + \int_1^2 x \left| \frac{y^2}{2} \right|_x^{\sqrt{2x}} dx \\
&= \frac{1}{2} \int_0^1 x(2x - 2x + x^2)\, dx + \frac{1}{2} \int_1^2 x(2x - x^2)\, dx \\
&= \frac{1}{2} \left| \frac{x^4}{4} \right|_0^1 + \frac{1}{2} \left| \frac{2x^3}{3} - \frac{x^4}{4} \right|_1^2 \\
&= \frac{1}{8} + \frac{8}{3} - 2 - \frac{1}{3} + \frac{1}{8} \\
&= \frac{7}{12}.
\end{aligned}
$$

## Example 15

*Evaluate* $\iint x^2 dxdy$, *over the region in the first quadrant enclosed by the rectangular hyperbola* $xy = 16$, *the lines* $y = x$, $y = 0$ *and* $x = 8$.

### Solution

1. The region of integration is $OPQR$.
2. (i) The points of intersection of the hyperbola $xy = 16$ and the line $y = x$ are obtained as

$$x^2 = 16, x = \pm 4$$

$$\therefore y = \pm 4$$

Hence, $R(4, 4)$ is the point of intersection in the first quadrant.

(ii) The point of intersection of the hyperbola $xy = 16$ and line $x = 8$ is obtained as

$$8y = 16$$

$$y = 2$$

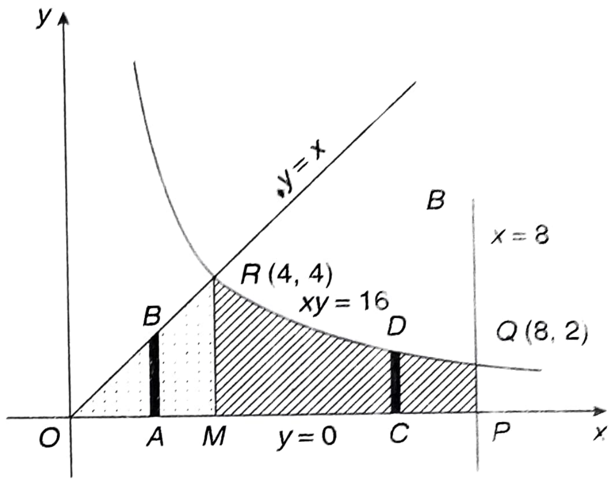


Fig. 5.18

The point of intersection is $Q(8, 2)$.

3. The integration can be done w.r.t. any variable first. To integrate w.r.t. $y$ first we need to draw a vertical strip in the region. But here one vertical strip cannot cover the entire region, and therefore divide the region $OPQR$ into two subregions $OMR$ and $RMPQ$ and draw one vertical strip in each subregion.
4. In the subregion $OMR$, strip starts from $x$ axis and terminates on the line $y = x$.
   Limits of $y$ : $y = 0$ to $y = x$
   Limits of $x$ : $x = 0$ to $x = 4$
5. In subregion $RMPQ$, strip starts from $x$ axis and terminates on the rectangular hyperbola $xy = 16$
   Limits of $y$ : $y = 0$ to $y = \dfrac{16}{x}$
   Limits of $x$ : $x = 4$ to $x = 8$

$$\begin{aligned}
I &= \iint x^2 dx\,dy \\
&= \iint_{OMR} x^2 dx\,dy + \iint_{RMPQ} x^2 dx\,dy \\
&= \int_0^4 \int_0^x x^2 dy\,dx + \int_4^8 \int_0^{\frac{16}{x}} x^2\,dy\,dx \\
&= \int_0^4 x^2 \int_0^x dy\,dx + \int_4^8 x^2 \int_0^{\frac{16}{x}} dy\,dx \\
&= \int_0^4 x^2 \left| y \right|_0^x dx + \int_4^8 x^2 \left| y \right|_0^{\frac{16}{x}} dx \\
&= \int_0^4 x^3 dx + \int_4^8 x^2 \cdot \frac{16}{x} dx \\
&= \left| \frac{x^4}{4} \right|_0^4 + 16 \left| \frac{x^2}{2} \right|_4^8 \\
&= 64 + 8(64 - 16) \\
&= 448.
\end{aligned}$$

## Example 16

*Evaluate* $\iint \frac{dx\,dy}{x^4+y^2}$, *over the region bounded by the* $y \ge x^2$, $x \ge 1$.

### Solution

1. The region of integration is bounded by $y \ge x^2$ (the region inside the parabola $x^2 = y$) and $x \ge 1$ (the region on the right of line $x = 1$).
2. The point of intersection of $x^2 = y$ and $x = 1$ is obtained as $1 = y$.
   The point of intersection is $P(1, 1)$.
3. Here, it is easier to integrate w.r.t. $y$ first than $x$. Draw a vertical strip $AB$ parallel to $y$-axis in the region which starts from the parabola $x^2 = y$ and extends up to infinity.
4. Limits of $y$ : $y = x^2$ to $y \to \infty$
   Limits of $x$ : $x = 1$ to $x \to \infty$

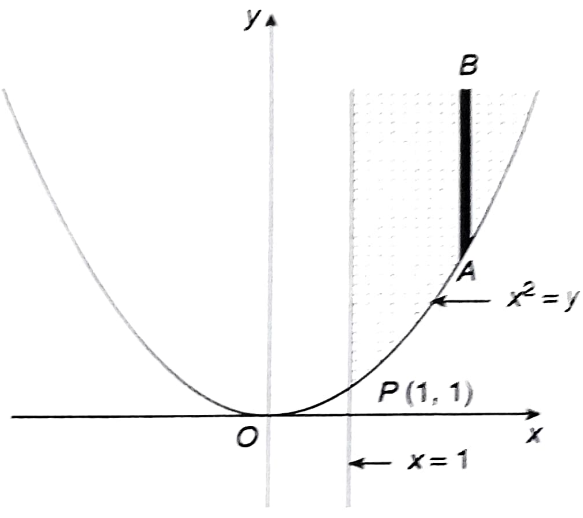


**Fig. 5.19**

$$I = \int_1^\infty \int_{x^2}^\infty \frac{1}{x^4+y^2}\,dy\,dx$$

$$= \int_1^\infty \left| \frac{1}{x^2} \tan^{-1} \frac{y}{x^2} \right|_{x^2}^\infty dx$$

$$= \int_1^\infty \frac{1}{x^2} (\tan^{-1} \infty - \tan^{-1} 1)\,dx$$

$$= \int_1^\infty \frac{1}{x^2} \left( \frac{\pi}{2} - \frac{\pi}{4} \right) dx$$

$$= \frac{\pi}{4} \left| -\frac{1}{x} \right|_1^\infty$$

$$= \frac{\pi}{4}.$$

## EXERCISE 5.2

Evaluate the following:

1. $\iint \frac{1}{xy}\,dx\,dy$, over the rectangle $1 \le x \le 2$, $1 \le y \le 2$.

   [Ans.: $(\log 2)^2$]

2. $\iint \sin \pi(ax + by)\,dx\,dy$, over the triangle bounded by the lines $x = 0$, $y = 0$ and $ax + by = 1$.

   $\left[ \text{Ans.: } \frac{1}{\pi ab} \right]$

3. $\iint e^{3x+4y}dx\,dy$, over the triangle bounded by the lines $x = 0$, $y = 0$, and $x + y = 1$.

$$\left[\text{Ans.}: \frac{1}{12}(3e^4 - 4e^3 + 1)\right]$$

4. $\iint xy\sqrt{1-x-y}\,dx\,dy$, over the triangle bounded by $x = 0$, $y = 0$ and $x - y = 1$.

$$\left[\text{Ans.}: \frac{16}{945}\right]$$

5. $\iint \sqrt{xy-y^2}\,dx\,dy$, over the triangle having vertices (0, 0), (10, 1), (1, 1).

[Ans.: 6]

6. $\iint (x+y+a)dx\,dy$, over the region bounded by the circle $x^2 + y^2 = a^2$.

[Ans.: $\pi a^3$]

7. $\iint xydxdy$, over the region bounded by the x-axis, the line $y = 2x$ and the parabola $y = \dfrac{x^2}{4a}$.

$$\left[\text{Ans.}: \frac{2048}{3}a^4\right]$$

8. $\iint 5-2x-y\,(dx)\,dy$, over the region bounded by x-axis, the line $x + 2y = 3$ and the parabola $y^2 = x$.

$$\left[\text{Ans.}: \frac{217}{60}\right]$$

9. $\iint (4x^2-y^2)^{\frac{1}{2}}dx\,dy$, over the triangle bounded by x-axis, the line $y = x$ and $x = 1$.

$$\left[\text{Ans.}: \frac{1}{3}\left(\frac{\pi}{3} + \frac{\sqrt{3}}{2}\right)\right]$$

10. $\iint xy(x+y)dx\,dy$, over the region bounded by the parabola $y^2 = x$, $x^2 = y$.

$$\left[\text{Ans.}: \frac{3}{28}\right]$$

11. $\iint xy(x+y)dx\,dy$, over the region bounded by the curve $x^2 = y$ and the line $x = y$.

$$\left[\text{Ans.}: \frac{3}{56}\right]$$

12. $\iint xy(x-1)dx\,dy$, over the region bounded by the rectangular hyperbola $xy = 4$, the lines $y = 0$, $x = 1$, $x = 4$ and x-axis.

[Ans.: 8(3 − log 4)]