Functions of Several Variables

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Introduction

The Space $\mathbb{R}\times\mathbb{R}$: The space $\mathbb{R}\times\mathbb{R}$ or $\mathbb{R}^2$ is the set of all ordered pairs of the form (x,y) such that $x\in\mathbb{R}$ and $y\in\mathbb{R}$. That is.,
$\mathbb{R}\times\mathbb{R}=\{(x,y):x\in\mathbb{R}\ \ and\ \ y\in\mathbb{R}\}$. If $\hat{x}=(x_1,x_2)$ and $\hat{y}=(y_1,y_2)$ are any two vectors or points in $\mathbb{R}\times\mathbb{R}$ then

1. $\hat{x}+\hat{y}=(x_1+y_1,\ x_2+y_2)$
2. $\alpha\hat{x}=(\alpha x_1\ ,\ \alpha x_2)$ where $\alpha\in\mathbb{R}$
3. $\hat{x}\,.\,\hat{y}=x_1y_1+x_2y_2$
4. $\|\hat{x}\|=\sqrt{x_1^2+x_2^2}$
5. $d(\hat{x}\,,\hat{y})=\|\hat{x}-\hat{y}\|=\sqrt{(x_1-y_1)^2+(x_2-y_2)^2}$

If $\hat{x}=(x_1,x_2)$ and $\hat{y}=(y_1,y_2)$ are any two vectors or points in $\mathbb{R}\times\mathbb{R}$ then their *dot product* $\hat{x}\,.\,\hat{y}=x_1y_1+x_2y_2$ is a real number.

If $\hat{x}=(x_1,x_2)$ is any vector in $\mathbb{R}\times\mathbb{R}$ then its *norm* or *length* $\|\hat{x}\|=\sqrt{x_1^2+x_2^2}$ is a non- negative real number. The non- negative real number $d(\hat{x}\,,\hat{y})$ is called the distance between $\hat{x}=(x_1,x_2)$ and $\hat{y}=(y_1,y_2)$.

Definition: Let $\hat{x}_0=(a,b)\in\mathbb{R}\times\mathbb{R}$. An ε - *neighborhood* of $\hat{x}_0=(a,b)$ is defined to be the set of all points $\hat{x}=(x,y)$ in $\mathbb{R}\times\mathbb{R}$ such that $\|\hat{x}-\hat{x}_0\|=\sqrt{(x-a)^2+(y-b)^2}<\varepsilon$.

The space $\mathbb{R}^n$: The space $\mathbb{R}^n$ is the set of all ordered n-tuples of the form $\hat{x}=(x_1,x_2,x_2,\ldots,x_n)$ such that $x_j\in\mathbb{R}$, where $j=1,2,3,\ldots,n$. The space $\mathbb{R}^n$ is called an n-dimensional Euclidean space. If $\hat{x}=(x_1,x_2,x_2,\ldots,x_n)$ and $\hat{y}=(y_1,y_2,y_3,\ldots,y_n)$ are any two ordered n-tuples then

1. $\hat{x}+\hat{y}=(x_1+y_1,\ x_2+y_2\ ,x_3+y_3,\ldots,x_n+y_n)$
2. $\alpha\hat{x}=(\alpha x_1\ ,\ \alpha x_2,\alpha x_3,\ldots,\alpha x_n)$ where $\alpha\in\mathbb{R}$
3. $\hat{x}\,.\,\hat{y}=x_1y_1+x_2y_2+x_3y_3+\ldots+x_ny_n$
4. $\|\hat{x}\|=\sqrt{x_1^2+x_2^2+x_3^2+\ldots+x_n^2}$
5. $d(\hat{x}\,,\hat{y})=\|\hat{x}-\hat{y}\|=\sqrt{(x_1-y_1)^2+(x_2-y_2)^2+(x_3-y_3)^2+\ldots+(x_n-y_n)^2}$

If $\hat{x} = (x_1, x_2, x_2, \dots, x_n)$ and $\hat{y} = (y_1, y_2, y_3, \dots, y_n)$ are any two vectors or points in $\mathbb{R}^n$ then their *dot product* $\hat{x} \,.\, \hat{y} = x_1y_1 + x_2y_2 + x_3y_3 + \dots + x_ny_n$ is a real number.

If $\hat{x} = (x_1, x_2, x_2, \dots, x_n)$ is any vector in $\mathbb{R}^n$ then its *norm* or *length* $\|\hat{x}\| = \sqrt{x_1^2 + x_2^2 + x_3^2 + \dots + x_n^2}$ is a non- negative real number. The non-negative real number $d(\hat{x}, \hat{y})$ is called the distance between the n-tuples $\hat{x} = (x_1, x_2, x_2, \dots, x_n)$ and $\hat{y} = (y_1, y_2, y_3, \dots, y_n)$.

Properties of norm: if $\hat{x} = (x_1, x_2, x_2, \dots, x_n)$ is any vector in $\mathbb{R}^n$ then its *norm* $\|\hat{x}\|$ is defined by $\|\hat{x}\| = \sqrt{x_1^2 + x_2^2 + x_3^2 + \dots + x_n^2}$. If $\hat{x} = (x_1, x_2, x_2, \dots, x_n)$ and $\hat{y} = (y_1, y_2, y_3, \dots, y_n)$ are any two ordered n-tuples in $\mathbb{R}^n$ then

1. $\|\hat{x}\| \geq 0$
2. $\|\hat{x}\| = 0 \Leftrightarrow \hat{x} = \hat{0}$
3. $\|\alpha\hat{x}\| = |\alpha| \, \|\hat{x}\|$ where $\alpha \in \mathbb{R}$
4. $\|\hat{x} + \hat{y}\| \leq \|\hat{x}\| + \|\hat{y}\|$
5. $\big|\|\hat{x}\| - \|\hat{y}\|\big| \leq \|\hat{x} - \hat{y}\|$

Definition: A function $f : \mathbb{R}^n \to \mathbb{R}^m$ is called a function of several variables. If $\hat{x} = (x_1, x_2, x_2, \dots, x_n)$ is an n-tuple in $\mathbb{R}^n$ and if $f(\hat{x}) = (y_1, y_2, \dots, y_m)$ then we define component functions $f_k : \mathbb{R}^n \to \mathbb{R}$ by $f_k(\hat{x}) = y_k$ where $k = 1, 2, 3, \dots, m$. Hence we write, $f(\hat{x}) = (f_1(\hat{x}), f_2(\hat{x}), \dots, f_m(\hat{x}))$ or $f = (f_1, f_2, \dots, f_m)$. The functions $f_1, f_2, \dots$ *and* f_m are called *component functions* of $f : \mathbb{R}^n \to \mathbb{R}^m$.

Examples

1. Define $f : \mathbb{R}^2 \to \mathbb{R}^2$ by $f(x, y) = (y, x)$. This is a function of two variables.
2. Define $V : \mathbb{R}^3 \to \mathbb{R}$ by $V(l, b, h) = lbh$, where l, b *and* h denote the length, breadth and height of a cuboid. Then $V(l, b, h)$ denotes its volume.
3. Let $S = \mathbb{R} \times \mathbb{R} - \{(0,0)\}$. Define $G : S \to \mathbb{R}$ by $G(h, w) = \dfrac{w}{h^2}$, where h and w denote the height in meters & weight in kgs of a person. The function $G(h, w) = \dfrac{w}{h^2}$ represents the BMI of a person.
4. Define $h : \mathbb{R}^3 \to \mathbb{R}^3$ by $h(x, y, z) = (xy, yz, zx)$.

Limits

Definition: Let $S \subseteq \mathbb{R} \times \mathbb{R}$ Suppose that $\hat{x}_0 = (a,b) \in \mathbb{R} \times \mathbb{R}$ is a limit point of S and $f : S \to \mathbb{R}$ is a function of two variables defined on S. We say that $\lim\limits_{(x,y)\to(a,b)} f(x,y) = L$ if for every $\varepsilon > 0$ there exists a $\delta > 0$ depending on $\varepsilon > 0$ and $\hat{x}_0$ such that $|f(x,y) - L| < \varepsilon$ for all $\hat{x} = (x,y) \in S$ for which $0 < \|\hat{x} - \hat{x}_0\| < \delta$. Here $L \in \mathbb{R}$.

Remark: If $f : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ and $\hat{x}_0 = (a,b) \in \mathbb{R} \times \mathbb{R}$ then there are three types of limits as follows.

1. $\lim\limits_{x\to a} \lim\limits_{y\to b} f(x,y)$
2. $\lim\limits_{y\to b} \lim\limits_{x\to a} f(x,y)$
3. $\lim\limits_{(x,y)\to(a,b)} f(x,y)$

Problem-1: Define $f : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ by $f(x,y) = \dfrac{2x^2y}{x^2+y^2+1}$. Find

a) $\lim\limits_{x\to 1} \lim\limits_{y\to 2} f(x,y)$

b) $\lim\limits_{y\to 2} \lim\limits_{x\to 1} f(x,y)$

Solution: Define $f : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ by $f(x,y) = \dfrac{2x^2y}{x^2+y^2+1}$.

a)
$$\lim_{x\to 1} \lim_{y\to 2} f(x,y) = \lim_{x\to 1} \lim_{y\to 2} \frac{2x^2y}{x^2+y^2+1}$$
$$= \lim_{x\to 1} \frac{4x^2}{x^2+5}$$
$$= \frac{2}{3}$$

b)
$$\lim_{y\to 2} \lim_{x\to 1} f(x,y) = \lim_{y\to 2} \lim_{x\to 1} \frac{2x^2y}{x^2+y^2+1}$$
$$= \lim_{y\to 1} \frac{2y}{y^2+2}$$
$$= \frac{2}{3}$$

Hence $\lim_{x\to1}\lim_{y\to2} f(x,y)=\lim_{y\to2}\lim_{x\to1} f(x,y)$.

Problem-2: Let $f(x,y)=\dfrac{x-y}{2x+y}$ where $2x+y\neq0$. Then find

a) $\lim_{x\to0}\lim_{y\to0} f(x,y)$

b) $\lim_{y\to0}\lim_{x\to0} f(x,y)$

Solution: Let $f(x,y)=\dfrac{x-y}{2x+y}$ where $2x+y\neq0$.

a) $$\begin{aligned}\lim_{x\to0}\lim_{y\to0} f(x,y)&=\lim_{x\to0}\lim_{y\to0}\frac{x-y}{2x+y}\\&=\lim_{x\to0}\frac{x}{2x}\\&=\frac{1}{2}\end{aligned}$$

b) $$\begin{aligned}\lim_{y\to0}\lim_{x\to0} f(x,y)&=\lim_{y\to0}\lim_{x\to0}\frac{x-y}{2x+y}\\&=\lim_{y\to0}\frac{-y}{y}\\&=-1\end{aligned}$$

Clearly, $\lim_{x\to0}\lim_{y\to0} f(x,y)\neq\lim_{y\to0}\lim_{x\to0} f(x,y)$.

Problem-3: Let $f(x,y)=\dfrac{x(y-1)}{y(x-1)}$ where $y\neq0$ and $x\neq1$. Then find

a) $\lim_{x\to1}\lim_{y\to1} f(x,y)$

b) $\lim_{y\to1}\lim_{x\to1} f(x,y)$

Solution: Let $f(x,y)=\dfrac{x(y-1)}{y(x-1)}$ where $y\neq0$ and $x\neq1$.

a) $\lim_{x\to1}\lim_{y\to1} f(x,y)=\lim_{x\to1}\lim_{y\to1}\dfrac{x(y-1)}{y(x-1)}=0$

b) $\lim_{y\to1}\lim_{x\to1} f(x,y)=\lim_{y\to1}\lim_{x\to1}\dfrac{x(y-1)}{y(x-1)}$ does not exist.

Problem-4: Let $f(x,y)=\dfrac{xy}{x^2+y^2}$ where $(x,y)\neq(0,0)$. Then find

a) $\lim\limits_{x\to0}\lim\limits_{y\to0} f(x,y)$

b) $\lim\limits_{y\to0}\lim\limits_{x\to0} f(x,y)$

Solution: Let $f(x,y)=\dfrac{xy}{x^2+y^2}$ where $(x,y)\neq(0,0)$.

a) $\lim\limits_{x\to0}\lim\limits_{y\to0} f(x,y)=\lim\limits_{x\to0}\lim\limits_{y\to0}\dfrac{xy}{x^2+y^2}=0$

b) $\lim\limits_{y\to0}\lim\limits_{x\to0} f(x,y)=\lim\limits_{y\to0}\lim\limits_{x\to0}\dfrac{xy}{x^2+y^2}=0$

Hence, $\lim\limits_{x\to0}\lim\limits_{y\to0} f(x,y)=\lim\limits_{y\to0}\lim\limits_{x\to0} f(x,y)$.

Problem-5: Define $f:\mathbb{R}\times\mathbb{R}\to\mathbb{R}$ by $f(x,y)=x^2+y^2+2$. Compute $\lim\limits_{(x,y)\to(0,0)} f(x,y)$.

Solution: Define $f:\mathbb{R}\times\mathbb{R}\to\mathbb{R}$ by $f(x,y)=x^2+y^2+2$. Let $\varepsilon>0$ be given. Choose $\delta=\sqrt{\dfrac{\varepsilon}{2}}$. Then, for all $\hat{x}=(x,y)\in\mathbb{R}\times\mathbb{R}$ for which $0<d\left(\hat{x},\hat{0}\right)<\delta$ or $0<\sqrt{x^2+y^2}<\delta$, we have $|f(x,y)-2|=|x^2+y^2|$

$$=x^2+y^2<\delta^2<\frac{\varepsilon}{2}<\varepsilon$$

$\Rightarrow |f(x,y)-2|<\varepsilon$

Hence $\lim\limits_{(x,y)\to(0,0)} f(x,y)=\lim\limits_{(x,y)\to(0,0)}\left(x^2+y^2+2\right)=2$

Problem-6: Let $f(x,y)=\dfrac{x^3+2x^2+xy^2+2y^2}{x^2+y^2}$, where $(x,y)\neq(0,0)$. Compute $\lim\limits_{(x,y)\to(0,0)} f(x,y)$.

Solution: Let $f(x,y)=\dfrac{x^3+2x^2+xy^2+2y^2}{x^2+y^2}$. Then

$$\lim_{(x,y)\to(0,0)} f(x,y)=\lim_{(x,y)\to(0,0)}\frac{x^3+2x^2+xy^2+2y^2}{x^2+y^2}$$

$$=\lim_{(x,y)\to(0,0)}\frac{x^2(x+2)+y^2(x+2)}{x^2+y^2}$$

$$= \lim_{(x,y)\to(0,0)} \frac{(x^2+y^2)(x+2)}{x^2+y^2}$$

$$= \lim_{(x,y)\to(0,0)} (x+2) = 2$$

$$\Rightarrow \lim_{(x,y)\to(0,0)} \frac{x^3+2x^2+xy^2+2y^2}{x^2+y^2} = 2.$$

Problem-7: Let $f(x,y) = \dfrac{2x^2y}{x^2+y^2}$, where $(x,y) \neq (0,0)$. Then show that $\lim_{(x,y)\to(0,0)} f(x,y) = 0$.

Solution: Let $f(x,y) = \dfrac{2x^2y}{x^2+y^2}$, where $(x,y) \neq (0,0)$. Let $\varepsilon > 0$ be given.

Choose $\delta = \dfrac{\varepsilon}{2}$. Then, for all $\hat{x} = (x,y) \in \mathbb{R} \times \mathbb{R}$ for which $0 < d(\hat{x}, \hat{0}) < \delta$ or or $0 < \sqrt{x^2+y^2} < \delta$, we have $|f(x,y) - 0| = |f(x,y)|$

$$= \left|\frac{2x^2y}{x^2+y^2}\right|$$

$$= \frac{2x^2|y|}{x^2+y^2}$$

$$\leq \frac{2x^2|y|}{x^2}$$

$$\leq 2|y|$$

$$\leq 2\sqrt{x^2+y^2}$$

$$\leq 2\,\delta = \varepsilon$$

$$\Rightarrow |f(x,y) - 0| < \varepsilon$$

Hence $\lim_{(x,y)\to(0,0)} f(x,y) = 0$.

Continuity

Definition: Let $S \subseteq \mathbb{R} \times \mathbb{R}$ and $\hat{x}_0 = (a,b) \in S$ and $f : S \to \mathbb{R}$ is a function of two variables defined on S. A function $f : S \to \mathbb{R}$ is said to be *continuous* at $\hat{x}_0 = (a,b)$ if for every $\varepsilon > 0$ there exists a $\delta > 0$ depending on $\varepsilon > 0$ and $\hat{x}_0$ such that $|f(x,y) - f(a,b)| < \varepsilon$ for all $\hat{x} = (x,y) \in S$ for which $\|\hat{x} - \hat{x}_0\| < \delta$.

Remarks

1. f is continuous at $\hat{x}_0=(a,b) \;\Leftrightarrow\; \lim\limits_{(x,y)\to(a,b)} f(x,y)=f(a,b)$
2. If f is not continuous at a point $\hat{x}_0$ then we say that f is discontinuous at $\hat{x}_0$. The point $\hat{x}_0$ is called a point of discontinuity of f.

Problem-1: Show that $f(x,y)=\begin{cases} \dfrac{x^2y}{x^2+y^2} & if \quad (x,y)\neq(0,0) \\ 0 & if \quad (x,y)=(0,0) \end{cases}$ is continuous at the origin.

Solution: Given $f(x,y)=\begin{cases} \dfrac{x^2y}{x^2+y^2} & if \quad (x,y)\neq(0,0) \\ 0 & if \quad (x,y)=(0,0) \end{cases}$

$\Rightarrow \quad f(0,0)=0$

If x and y are any two real numbers, then $(x-y)\geq 0$

$\Rightarrow \; x^2+y^2-2xy\geq 0$

$\Rightarrow \; x^2+y^2\geq 2xy$

$\Rightarrow \; xy\leq \dfrac{x^2+y^2}{2}$ and $|xy|\leq \dfrac{x^2+y^2}{2} \qquad \to (1)$

Let $\varepsilon>0$ be given and choose $\delta=2\varepsilon$.

For any all $\hat{x}=(x,y)\in\mathbb{R}\times\mathbb{R}$ with $d(\hat{x},\hat{0})=\|\hat{x}-\hat{0}\|=\sqrt{x^2+y^2}<\delta$, we have

$$\begin{aligned}
|f(x,y)-f(0,0)| &= \left|\frac{x^2y}{x^2+y^2}-0\right| \\
&= \left|\frac{x^2y}{x^2+y^2}\right| \\
&= \frac{|x||xy|}{x^2+y^2} \\
&\leq \left(\frac{|x|}{x^2+y^2}\right)\left(\frac{x^2+y^2}{2}\right) \\
&= \frac{|x|}{2}
\end{aligned}$$

$= \dfrac{\sqrt{x^2+y^2}}{2}$

$< \dfrac{\delta}{2} = \varepsilon$

$\Rightarrow |f(x,y) - f(0,0)| < \varepsilon$

Thus for every $\varepsilon > 0$ there exists a $\delta > 0$ such that $|f(x,y) - f(0,0)| < \varepsilon$ for all $(x,y) \in \mathbb{R} \times \mathbb{R}$ for which $\sqrt{x^2+y^2} < \delta$.

$\Rightarrow f$ is continuous at the origin.

Problem-2: Show that $f(x,y) = \begin{cases} \dfrac{2xy}{x^2+y^2} & if \quad (x,y) \neq (0,0) \\ 0 & if \quad (x,y) = (0,0) \end{cases}$

is discontinuous at $(0,0)$.

Solution: Given $f(x,y) = \begin{cases} \dfrac{2xy}{x^2+y^2} & if \quad (x,y) \neq (0,0) \\ 0 & if \quad (x,y) = (0,0) \end{cases}$

$\Rightarrow \quad f(0,0) = 0$

Assume that f is continuous at $(0,0)$. Then for $\varepsilon = \dfrac{1}{2}$ there exists a $\delta > 0$ such that $|f(x,y) - f(0,0)| < \varepsilon$ for all $(x,y) \in \mathbb{R} \times \mathbb{R}$ for which $\sqrt{x^2+y^2} < \delta$.

$\Rightarrow \left| \dfrac{2xy}{x^2+y^2} \right| < \dfrac{1}{2}$ for all $(x,y) \in \mathbb{R} \times \mathbb{R}$ for which $\sqrt{x^2+y^2} < \delta$

Choose a point (u,v) on the line $y = x$ and inside the circle centred at the origin with radius δ

$\Rightarrow v = u$ and $\sqrt{u^2+v^2} < \delta$

$\Rightarrow \left| \dfrac{2uv}{u^2+v^2} \right| < \dfrac{1}{2}$

$\Rightarrow \left| \dfrac{2u^2}{u^2+u^2} \right| < \dfrac{1}{2}$

$\Rightarrow 1 < \dfrac{1}{2}$ which is a contradiction

Hence f is discontinuous at the origin.

Exercise

1. If $f(x,y)=\dfrac{x-y}{x+y}$, $x+y\neq 0$ the find

   a) $\lim\limits_{x\to 0}\lim\limits_{y\to 0} f(x,y)$

   b) $\lim\limits_{y\to 0}\lim\limits_{x\to 0} f(x,y)$

2. Show that $f(x,y)=\dfrac{x^2y^2}{x^2y^2+(x-y)^2}$ whenever $x^2y^2+(x-y)^2\neq 0$. Show that $\lim\limits_{x\to 0}\lim\limits_{y\to 0} f(x,y)=\lim\limits_{y\to 0}\lim\limits_{x\to 0} f(x,y)=0$.

3. Let $f(x,y)=\begin{cases} x\sin\left(\dfrac{1}{y}\right) & if \quad y\neq 0 \\ 0 & if \quad y=0 \end{cases}$

   Then find

   a) $\lim\limits_{x\to 0}\lim\limits_{y\to 0} f(x,y)$

   b) $\lim\limits_{y\to 0}\lim\limits_{x\to 0} f(x,y)$

   c) $\lim\limits_{(x,y)\to(0,0)} f(x,y)$

4. If $f(x,y)=\dfrac{x^2-y^2}{x^2+y^2}$, $(x,y)\neq(0,0)$ then find $\lim\limits_{(x,y)\to(0,0)} f(x,y)$ on the straight line $y=mx$.

5. Discuss the continuity of $f(x,y)=\begin{cases} 0 & if \quad y\leq 0 \;\; or \; y\geq x^2 \\ 1 & if \quad 0<y<x^2 \end{cases}$ at origin.

6. Show that $f(x,y)=\dfrac{x+y}{x-y}$, $x\neq y$ is continuous at $(1,2)$.

7. Let $f(x,y)=\begin{cases} \dfrac{x^4-y^4}{x^4+y^4} & if \quad (x,y)\neq(0,0) \\ 0 & if \quad (x,y)=(0,0) \end{cases}$

   Show that $f(x,y)$ is not continuous at the origin.

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Partial Derivatives

Definition: Let $S \subseteq \mathbb{R} \times \mathbb{R}$. *The partial Derivatives* of a function $f : S \to \mathbb{R}$ are defined as follows.

$$\frac{\partial f}{\partial x} = f_x(x,y) = \lim_{h\to 0} \frac{f(x+h,y)-f(x,y)}{h}$$

$$\frac{\partial f}{\partial y} = f_y(x,y) = \lim_{k\to 0} \frac{f(x,y+k)-f(x,y)}{k}$$

$$\frac{\partial^2 f}{\partial x^2} = f_{xx}(x,y) = \lim_{h\to 0} \frac{f_x(x+h,y)-f_x(x,y)}{h}$$

$$\frac{\partial^2 f}{\partial y^2} = f_{yy}(x,y) = \lim_{k\to 0} \frac{f_y(x,y+k)-f_y(x,y)}{k}$$

$$\frac{\partial^2 f}{\partial x \partial y} = f_{xy}(x,y) = \lim_{h\to 0} \frac{f_y(x+h,y)-f_y(x,y)}{h}$$

$$\frac{\partial^2 f}{\partial y \partial x} = f_{yx}(x,y) = \lim_{k\to 0} \frac{f_x(x,y+k)-f_x(x,y)}{k}$$

Remark:

1. Let $S \subseteq \mathbb{R} \times \mathbb{R}$. If $f : S \to \mathbb{R}$ is continuous and if either f_x or f_y is continuous on S then $\frac{\partial^2 f}{\partial x \partial y} = \frac{\partial^2 f}{\partial y \partial x}$.
2. In general, $\frac{\partial^2 f}{\partial x \partial y} \neq \frac{\partial^2 f}{\partial y \partial x}$.

Example: Define $f : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ by $f(x,y) = \frac{xy\left(x^2-y^2\right)}{x^2+y^2}$ for $(x,y) \neq (0,0)$ and $f(0,0)=0$. Then $\frac{\partial^2 f}{\partial x \partial y} = 1$ at $(0,0)$

$$\frac{\partial^2 f}{\partial y \partial x} = -1 \quad \text{at } (0,0)$$

Hence $\frac{\partial^2 f}{\partial x \partial y} \neq \frac{\partial^2 f}{\partial y \partial x}$ at $(0,0)$.

Definition: Let $S \subseteq \mathbb{R}^3$. A function $f : S \to \mathbb{R}$ are defined as follows.

$$\frac{\partial f}{\partial x} = f_x(x,y,z) = \lim_{h\to 0} \frac{f(x+h,y,z)-f(x,y,z)}{h}$$

$$\frac{\partial f}{\partial y} = f_y(x,y,z) = \lim_{k\to 0}\frac{f(x,y+k,z)-f(x,y,z)}{k}$$

$$\frac{\partial f}{\partial z} = f_z(x,y,z) = \lim_{j\to 0}\frac{f(x,y,z+j)-f(x,y,z)}{j}$$

Problem-1: Find the first and second order partial derivatives for the function $f(x,y)=\sin^{-1}\left(\frac{x}{y}\right)$.

Solution: Given $f(x,y)=\sin^{-1}\left(\frac{x}{y}\right)$. Then

$$\frac{\partial f}{\partial x} = \frac{\partial}{\partial x}\sin^{-1}\left(\frac{x}{y}\right) = \frac{1}{\sqrt{1-\left(\frac{x}{y}\right)^2}}\frac{1}{y} = \frac{1}{\sqrt{y^2-x^2}}$$

$$\frac{\partial f}{\partial y} = \frac{\partial}{\partial y}\sin^{-1}\left(\frac{x}{y}\right) = \frac{1}{\sqrt{1-\left(\frac{x}{y}\right)^2}}\left(\frac{-x}{y^2}\right) = \frac{-x}{y\sqrt{y^2-x^2}}$$

$$\frac{\partial^2 f}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}\right) = \frac{\partial}{\partial x}\frac{1}{\sqrt{y^2-x^2}} = \frac{-\left(y^2-x^2\right)^{-3/2}}{2}(-2x) = \frac{x}{\left(y^2-x^2\right)^{3/2}}$$

$$\frac{\partial^2 f}{\partial y^2} = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial y}\right) = \frac{\partial}{\partial y}\frac{-x}{y\sqrt{y^2-x^2}} = -x\left[\left(\frac{1}{y}\right)\left(\frac{-1}{2}\right)\left(y^2-x^2\right)^{-3/2}2y+\left(y^2-x^2\right)^{-3/2}\left(\frac{-1}{y^2}\right)\right]$$

$$= -x\left[-\left(y^2-x^2\right)^{-3/2}-\frac{\left(y^2-x^2\right)^{-3/2}}{y^2}\right]$$

$$= \frac{x}{\left(y^2-x^2\right)^{3/2}}+\frac{1}{y^2\left(y^2-x^2\right)^{3/2}}$$

$$\frac{\partial^2 f}{\partial x\partial y} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right) = \frac{\partial}{\partial x}\left(\frac{-x}{y\sqrt{y^2-x^2}}\right) = -\frac{1}{y}\frac{\partial}{\partial x}\left(\frac{x}{\sqrt{y^2-x^2}}\right)$$

$$= -\frac{1}{y}\left(\frac{\sqrt{y^2-x^2} \; - x\,\dfrac{1}{2\sqrt{y^2-x^2}}\,(-2x)}{y^2-x^2}\right)$$

$$= -\frac{1}{y}\left(\frac{\sqrt{y^2-x^2} \; + \dfrac{x^2}{\sqrt{y^2-x^2}}}{y^2-x^2}\right)$$

$$= -\frac{1}{y}\left(\frac{y^2}{\left(y^2-x^2\right)\sqrt{y^2-x^2}}\right)$$

$$= -\frac{y}{\left(y^2-x^2\right)^{3/2}}$$

Clearly, $\dfrac{\partial^2 f}{\partial x \partial y} = \dfrac{\partial^2 f}{\partial y \partial x}$.

Problem-2: If $u(x,y) = \tan^{-1}\left(\dfrac{2xy}{x^2-y^2}\right)$ then prove that $\dfrac{\partial^2 u}{\partial x^2} + \dfrac{\partial^2 u}{\partial y^2} = 0$.

Solution: Given $u(x,y) = \tan^{-1}\left(\dfrac{2xy}{x^2-y^2}\right)$

$$\Rightarrow \frac{\partial u}{\partial x} = \frac{\partial}{\partial x}\tan^{-1}\left(\frac{2xy}{x^2-y^2}\right)$$

$$= \frac{1}{1+\left(\dfrac{2xy}{x^2-y^2}\right)^2}\frac{\partial}{\partial x}\left(\frac{2xy}{x^2-y^2}\right)$$

$$= \frac{\left(x^2-y^2\right)^2}{\left(x^2-y^2\right)^2+4x^2y^2}\left[\frac{2\left(x^2-y^2\right)y-4x^2y}{\left(x^2-y^2\right)^2}\right]$$

$$= \frac{2\left(x^2-y^2\right)y-4x^2y}{\left(x^2-y^2\right)^2+4x^2y^2}$$

$$= \frac{-4x^2y-2y^3}{\left(x^2-y^2\right)^2+4x^2y^2}$$

$$= \frac{-2y\left(x^2+y^2\right)}{\left(x^2+y^2\right)^2}$$

$$= \frac{-2y}{x^2+y^2}$$

$$\Rightarrow \frac{\partial^2 u}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{-2y}{x^2+y^2}\right)$$

$$= (-2y)\frac{\partial}{\partial x}\frac{1}{x^2+y^2}$$

$$= (-2y)\left(\frac{-2x}{\left(x^2+y^2\right)^2}\right)$$

$$= \frac{4xy}{\left(x^2+y^2\right)^2}$$

$$\Rightarrow \frac{\partial^2 u}{\partial x^2} = \frac{4xy}{\left(x^2+y^2\right)^2} \quad \rightarrow (1)$$

Similarly, $\dfrac{\partial u}{\partial y} = \dfrac{\partial}{\partial y}\tan^{-1}\left(\dfrac{2xy}{x^2-y^2}\right)$

$$= \frac{1}{1+\left(\frac{2xy}{x^2-y^2}\right)^2}\frac{\partial}{\partial y}\left(\frac{2xy}{x^2-y^2}\right)$$

$$= \frac{\left(x^2-y^2\right)^2}{\left(x^2-y^2\right)^2+4x^2y^2}\left[\frac{2\left(x^2-y^2\right)x-2xy(-2y)}{\left(x^2-y^2\right)^2}\right]$$

$$= \frac{2x^3+2xy^2}{\left(x^2-y^2\right)^2+4x^2y^2}$$

$$= \frac{2x\left(x^2+y^2\right)}{\left(x^2+y^2\right)^2}$$

$$= \frac{2x}{x^2+y^2}$$

$$\Rightarrow \quad \frac{\partial^2 u}{\partial y^2} = \frac{\partial}{\partial y}\left(\frac{2x}{x^2+y^2}\right)$$

$$= 2x\frac{\partial}{\partial x}\left(\frac{1}{x^2+y^2}\right)$$

$$= 2x\left(\frac{-2y}{\left(x^2+y^2\right)^2}\right)$$

$$= \frac{-4xy}{\left(x^2+y^2\right)^2}$$

$$\Rightarrow \quad \frac{\partial^2 u}{\partial y^2} = \frac{-4xy}{\left(x^2+y^2\right)^2} \quad \rightarrow (2)$$

From (1) and (2), we have $\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}=0$.

Problem-3: If $z(x+y)=x^2+y^2$ then prove that $\left(\frac{\partial z}{\partial x}-\frac{\partial z}{\partial y}\right)^2=4\left(1-\frac{\partial z}{\partial x}-\frac{\partial z}{\partial y}\right)$.

Solution: Let $z=f(x,y)$ and given that $z(x+y)=x^2+y^2$

$$\Rightarrow \quad z=f(x,y)=\frac{x^2+y^2}{x+y}$$

$$\Rightarrow \quad \frac{\partial z}{\partial x} = \frac{(x+y)2x-\left(x^2+y^2\right)}{(x+y)^2}$$

$$= \frac{x^2+2xy-y^2}{(x+y)^2}$$

$$\Rightarrow \quad \frac{\partial z}{\partial x} = \frac{x^2+2xy-y^2}{(x+y)^2} \quad \rightarrow (1)$$

Similarly, $\frac{\partial z}{\partial y} = \frac{(x+y)2y-\left(x^2+y^2\right)}{(x+y)^2}$

$$= \frac{-x^2+2xy+y^2}{(x+y)^2}$$

$$\Rightarrow \frac{\partial z}{\partial y} = \frac{-x^2+2xy+y^2}{(x+y)^2} \quad \rightarrow (2)$$

Now, $$\left(\frac{\partial z}{\partial x} - \frac{\partial z}{\partial y}\right)^2 = \left(\frac{x^2+2xy-y^2-\left(-x^2+2xy+y^2\right)}{(x+y)^2}\right)^2$$

$$= \left(\frac{2x^2-2y^2}{(x+y)^2}\right)^2$$

$$= \left(\frac{2\left(x^2-y^2\right)}{(x+y)^2}\right)^2$$

$$= 4\,\frac{(x+y)^2(x-y)^2}{(x+y)^4}$$

$$= \frac{4(x-y)^2}{(x+y)^2}$$

$$\Rightarrow \left(\frac{\partial z}{\partial x} - \frac{\partial z}{\partial y}\right)^2 = \frac{4(x-y)^2}{(x+y)^2} \quad \rightarrow (3)$$

Now, $$4\left(1-\frac{\partial z}{\partial x}-\frac{\partial z}{\partial y}\right) = 4\left(1-\frac{x^2+2xy-y^2}{(x+y)^2} - \frac{-x^2+2xy+y^2}{(x+y)^2}\right)$$

$$= 4\left(\frac{(x+y)^2-\left(x^2+2xy-y^2\right)-\left(-x^2+2xy+y^2\right)}{(x+y)^2}\right)$$

$$= 4\left(\frac{(x+y)^2-4xy}{(x+y)^2}\right)$$

$$= 4\,\frac{(x-y)^2}{(x+y)^2}$$

$$\Rightarrow 4\left(1-\frac{\partial z}{\partial x}-\frac{\partial z}{\partial y}\right) = 4\,\frac{(x-y)^2}{(x+y)^2} \quad \rightarrow (4)$$

From (3) and (4)

$$\left(\frac{\partial z}{\partial x} - \frac{\partial z}{\partial y}\right)^2 = 4\left(1-\frac{\partial z}{\partial x}-\frac{\partial z}{\partial y}\right).$$

Problem-4: If $u = \log\left(x^3 + y^3 + z^3 - 3xyz\right)$ then show that

a) $\left(\frac{\partial}{\partial x} + \frac{\partial}{\partial y} + \frac{\partial}{\partial z}\right)^2 u = \frac{-9}{(x+y+z)^2}$

b) $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial x^2} = \frac{-3}{(x+y+z)^2}$

Solution: Given that $u = \log\left(x^3 + y^3 + z^3 - 3xyz\right)$

$$\Rightarrow \quad \frac{\partial u}{\partial x} = \frac{3\left(x^2 - yz\right)}{x^3 + y^3 + z^3 - 3xyz}, \quad \frac{\partial u}{\partial y} = \frac{3\left(y^2 - xz\right)}{x^3 + y^3 + z^3 - 3xyz},$$

$$\frac{\partial u}{\partial z} = \frac{3\left(z^2 - xy\right)}{x^3 + y^3 + z^3 - 3xyz}$$

$$\Rightarrow \frac{\partial^2 u}{\partial x^2} = \frac{-3\left(x^4 - 2xy^3 - 2xz^3 + 3y^2z^2\right)}{\left(x^3 + y^3 + z^3 - 3xyz\right)^2}, \qquad \frac{\partial^2 u}{\partial y^2} = \frac{-3\left(y^4 - 2yz^3 - 2yx^3 + 3x^2z^2\right)}{\left(x^3 + y^3 + z^3 - 3xyz\right)^2}$$

$$\frac{\partial^2 u}{\partial z^2} = \frac{-3\left(z^4 - 2zx^3 - 2xy^3 + 3x^2y^2\right)}{\left(x^3 + y^3 + z^3 - 3xyz\right)^2}$$

a)
$$\begin{aligned}
\left(\frac{\partial}{\partial x} + \frac{\partial}{\partial y} + \frac{\partial}{\partial z}\right)^2 u &= \left(\frac{\partial}{\partial x} + \frac{\partial}{\partial y} + \frac{\partial}{\partial z}\right)\left(\frac{\partial}{\partial x} + \frac{\partial}{\partial y} + \frac{\partial}{\partial z}\right) u \\
&= \left(\frac{\partial}{\partial x} + \frac{\partial}{\partial y} + \frac{\partial}{\partial z}\right)\left(\frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} + \frac{\partial u}{\partial z}\right) \\
&= \left(\frac{\partial}{\partial x} + \frac{\partial}{\partial y} + \frac{\partial}{\partial z}\right)\left(\frac{3\left(x^2 + y^2 + z^2 - xy - yz - zx\right)}{x^3 + y^3 + z^3 - 3xyz}\right) \\
&= \left(\frac{\partial}{\partial x} + \frac{\partial}{\partial y} + \frac{\partial}{\partial z}\right)\left(\frac{3}{x+y+z}\right) \\
&= \frac{-3}{(x+y+z)^2} + \frac{-3}{(x+y+z)^2} + \frac{-3}{(x+y+z)^2} \\
&= \frac{-9}{(x+y+z)^2}
\end{aligned}$$

$$\Rightarrow \left(\frac{\partial}{\partial x} + \frac{\partial}{\partial y} + \frac{\partial}{\partial z}\right)^2 u = \frac{-9}{(x+y+z)^2}$$

b) $\dfrac{\partial^2 u}{\partial x^2}+\dfrac{\partial^2 u}{\partial x^2}+\dfrac{\partial^2 u}{\partial x^2}=\dfrac{-3\left(x^2+y^2+z^2-xy-yz-zx\right)^2}{\left(x^3+y^3+z^3-3xyz\right)}$

$$=\frac{-3}{(x+y+z)^2}$$

$$\Rightarrow \frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial x^2}=\frac{-3}{(x+y+z)^2}.$$

Euler's theorem

Definition: Let $S\subseteq \mathbb{R}\times\mathbb{R}$. A function $f:S\to\mathbb{R}$ is said to be a *homogeneous function of degree* k if $f(\lambda x,\lambda y)=\lambda^k f(x,y)$ for all $(x,y)\in S$. A function $f:S\to\mathbb{R}$ is said to be a *homogeneous function* if $f(\lambda x,\lambda y)=f(x,y)$ for all $(x,y)\in S$.

Examples:

1. The function $f(x,y)=\dfrac{x-y}{x+y}$, $(x,y)\neq(0,0)$ is a homogeneous function of degree zero.
2. The function $f(x,y)=\dfrac{x^2-y^2}{x+y}$, $(x,y)\neq(0,0)$ is a homogeneous function of degree one.

Euler's theorem: If $z=f(x,y)$ is a homogeneous function of degree n, then $x\dfrac{\partial z}{\partial x}+y\dfrac{\partial z}{\partial y}=nz$, $\forall\, x,y$ in the domain of the function.

Proof: Suppose that $z=f(x,y)$ is a homogeneous function of degree n.

$$\Rightarrow z=f(x,y)=f\left(x,\frac{xy}{x}\right)=x^n f\left(1,\frac{y}{x}\right)$$

$$\Rightarrow z=x^n g\left(\frac{y}{x}\right) \qquad \to (1) \quad \text{where} \quad g\left(\frac{y}{x}\right)=f\left(1,\frac{y}{x}\right)$$

$$\Rightarrow \frac{\partial z}{\partial x}=x^n g'\left(\frac{y}{x}\right)\left(\frac{-y}{x^2}\right)+n\,x^{n-1}g\left(\frac{y}{x}\right)$$

$$\Rightarrow \frac{\partial z}{\partial x}=-\,x^{n-2}\,y\,g'\left(\frac{y}{x}\right)+n\,x^{n-1}g\left(\frac{y}{x}\right)$$

$$\Rightarrow \; x\frac{\partial z}{\partial x} = -\, x^{n-2}\; y\; g'\left(\frac{y}{x}\right) \; + n\; x^{n} g\left(\frac{y}{x}\right)$$

$$\Rightarrow \; x\frac{\partial z}{\partial x} = -\, x^{n-2}\; y\; g'\left(\frac{y}{x}\right) \; + nz \qquad \rightarrow \; (2)$$

Similarly, $\dfrac{\partial z}{\partial y} = \; x^{n} g'\left(\dfrac{y}{x}\right)\left(\dfrac{1}{x}\right) = x^{n-1}\; g'\left(\dfrac{y}{x}\right)$

$$\Rightarrow \; y\frac{\partial z}{\partial y} = x^{n-1}\; y\; g'\left(\frac{y}{x}\right) \qquad \rightarrow \; (3)$$

Adding equations (2) and (3)

$$x\,\frac{\partial z}{\partial x} + y\,\frac{\partial z}{\partial y} = nz\,.$$

Problem-1: Verify Euler's theorem for the function $z = ax^2 + 2hxy + by^2$, where $a > 0$ and $b > 0$.

Solution: Given $z = f(x,y) = ax^2 + 2hxy + by^2$, where $a > 0$ and $b > 0$. Then

$$f(\lambda x, \lambda y) \; = \; a(\lambda x)^2 + 2h(\lambda x)(\lambda y) + b(\lambda y)^2$$

$$= \; \lambda^2 \left(ax^2 + 2hxy + by^2\right)$$

$$= \; \lambda^2 \; f(x,y)$$

$$\Rightarrow \; f(\lambda x, \lambda y) = \; \lambda^2 \; f(x,y) \;\; \forall \; x,y$$

$\Rightarrow \; z = ax^2 + 2hxy + by^2$ is a homogeneous function of degree 2.

Consider $z = ax^2 + 2hxy + by^2 \;\Rightarrow\; \dfrac{\partial z}{\partial x} = 2ax + 2hy$ and $\dfrac{\partial z}{\partial y} = 2hx + 2by$

$$\Rightarrow \;\; x\,\frac{\partial z}{\partial x} \; = 2ax^2 + 2hxy \quad \text{and} \quad y\,\frac{\partial z}{\partial y} \; = 2hxy + 2by^2$$

$$\Rightarrow \;\; x\,\frac{\partial z}{\partial x} \; + y\,\frac{\partial z}{\partial y} \; = \; 2ax^2 + 2hxy + 2hxy + 2by^2$$

$$= \; 2ax^2 + 4hxy + 2by^2$$

$$= \; 2\left(ax^2 + 2hxy + 2by^2\right)$$

$$= \; 2z$$

$$\Rightarrow \quad x\frac{\partial z}{\partial x} + y\frac{\partial z}{\partial y} = 2z$$

Hence the Euler's theorem is verified.

Exercise

1. Find all partial derivatives of the following

   a) $\log\left(x^2+y^2\right)$ b) $\tan\left(\tan^{-1}x+\tan^{-1}y\right)$ c) e^{x^y}

2. If $u=\log\left(x^2+y^2+z^2\right)$ prove that $x\dfrac{\partial^2 u}{\partial y\,\partial z} = y\dfrac{\partial^2 u}{\partial z\,\partial x} = z\dfrac{\partial^2 u}{\partial x\,\partial y}$.

3. If $f(x,y)=\log\left(x^2+y^2\right)+\tan^{-1}\left(\dfrac{y}{x}\right)$, then prove that $\dfrac{\partial^2 f}{\partial x^2}+\dfrac{\partial^2 f}{\partial y^2}=0$.

4. Verify Euler's theorem for the function $u=\sin^{-1}\left(\dfrac{x}{y}\right)+\tan^{-1}\left(\dfrac{y}{x}\right)$

5. If $u=\cot^{-1}\left(\dfrac{x+y}{\sqrt{x}+\sqrt{y}}\right)$ show that $x\dfrac{\partial u}{\partial x}+y\dfrac{\partial u}{\partial y}+\dfrac{1}{4}\sin 2u=0$.

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Total Derivatives

1. If $z = f(x,y)$ is a function of two variables then its total derivative is defines by
$$\frac{dz}{dt} = \frac{\partial z}{\partial x}\frac{dx}{dt} + \frac{\partial z}{\partial y}\frac{dz}{dt} \quad \text{or} \quad df = \frac{\partial f}{\partial x}dx + \frac{\partial f}{\partial y}dy$$
2. If $w = f(x,y,z)$ is a function of three variables then its total derivative is defined by
$$\frac{dw}{dt} = \frac{\partial w}{\partial x}\frac{dx}{dt} + \frac{\partial w}{\partial y}\frac{dy}{dt} + \frac{\partial w}{\partial z}\frac{dz}{dt} \quad \text{or } df = \frac{\partial f}{\partial x}dx + \frac{\partial f}{\partial y}dy + \frac{\partial f}{\partial z}dz$$

Problem-1: If $u(x,y) = x^2 + y^2$, $x = at^2$, $y = 2at$, where $a > 0$ then find $\frac{du}{dt}$.

Solution: Given that $u(x,y) = x^2 + y^2$, $x = at^2$, $y = 2at$, where $a > 0$

$$\Rightarrow \frac{\partial u}{\partial x} = 2x \,,\ \frac{\partial u}{\partial y} = 2y \,,\ \frac{dx}{dt} = 2at \text{ and } \frac{dy}{dt} = 2a$$

$$\Rightarrow \frac{du}{dt} = \frac{\partial u}{\partial x}\frac{dx}{dt} + \frac{\partial u}{\partial y}\frac{dz}{dt}$$

$$= (2x)(2at) + (2y)(2a)$$

$$= 4xat + 4ay$$

$$= 4a(xt + y)$$

$$= 4a(at^2t + 2at)$$

$$= 4a(at^3 + 2at)$$

$$= 4at^2(t + 2)$$

$$\Rightarrow \frac{du}{dt} = 4at^2(t + 2)$$

Problem-2: If $u(x,y,z) = x^2 + y^2 + z^2$, $x = e^t$, $y = e^t \sin t$ and $z = e^t \cos t$ then find $\frac{du}{dt}$.

Solution: Given that $u(x,y,z) = x^2 + y^2 + z^2$, $x = e^t$, $y = e^t \sin t$ and $z = e^t \cos t$

$\Rightarrow \dfrac{\partial u}{\partial x} = 2x,\ \dfrac{\partial u}{\partial y} = 2y\ ,\ \dfrac{\partial u}{\partial z} = 2z\ ,\ \dfrac{dx}{dt} = e^t,\ \dfrac{dy}{dt} = e^t(\cos t + \sin t)$ and

$\dfrac{dz}{dt} = e^t(\cos t - \sin t)$

$$\Rightarrow \frac{du}{dt} = \frac{\partial u}{\partial x}\frac{dx}{dt} + \frac{\partial u}{\partial y}\frac{dy}{dt} + \frac{\partial u}{\partial z}\frac{dz}{dt}$$

$$= (2x)e^t + (2y)\, e^t(\cos t + \sin t) + (2z)\, e^t(\cos t - \sin t)$$

$$= 2e^t\, e^t + 2e^t \sin t\, e^t(\cos t + \sin t) + 2e^t \cos t e^t(\cos t - \sin t)$$

$$= e^{2t}\left(2 + 2\sin t\,(\cos t + \sin t) + 2\cos t(\cos t - \sin t)\right)$$

$$= 4e^{2t}$$

$$\Rightarrow \frac{du}{dt} = 4e^{2t}.$$

Exercise

1. If $u = \sin\left(\dfrac{x}{y}\right),\ x = e^t\ ,\ y = t^2$ then find $\dfrac{du}{dt}$.
2. If $u = xy + yz + zx\ ,\ x = \dfrac{1}{t},\ y = e^t$ and $z = e^t$ then find $\dfrac{du}{dt}$.
3. If $u = f(y - z, z - x, x - y)$ prove that $\dfrac{\partial u}{\partial x} + \dfrac{\partial u}{\partial y} + \dfrac{\partial u}{\partial z} = 0$.

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Differentiability

Definition: A function $f:\mathbb{R}\times\mathbb{R}\to\mathbb{R}$ is said to be differentiable at a point (x_0,y_0) if

1. $\dfrac{\partial f}{\partial x}$ and $\dfrac{\partial f}{\partial y}$ exist at (x_0,y_0)
2. $$\lim_{(x,y)\to(x_0,y_0)} \frac{\left| f(x,y)-f(x_0,y_0)-\left[\dfrac{\partial f}{\partial x}(x_0,y_0)\right](x-x_0)\left[\dfrac{\partial f}{\partial y}(x_0,y_0)\right](y-y_0)\right|}{\left\|(x,y)-(x_0,y_0)\right\|} = 0$$

Definition: Let $F:\mathbb{R}\times\mathbb{R}\to\mathbb{R}\times\mathbb{R}$ be a function of two variables whose component functions are $f:\mathbb{R}\times\mathbb{R}\to\mathbb{R}$ and $g:\mathbb{R}\times\mathbb{R}\to\mathbb{R}$. Suppose that $\dfrac{\partial f}{\partial x},\dfrac{\partial f}{\partial x},\dfrac{\partial g}{\partial x}$ and $\dfrac{\partial g}{\partial y}$ exist at a point $\hat{x}_0=(x_0,y_0)$. Then the matrix

$$J_f(\hat{x}_0)=\begin{bmatrix} \dfrac{\partial f}{\partial x}(\hat{x}_0) & \dfrac{\partial f}{\partial y}(\hat{x}_0) \\ \dfrac{\partial g}{\partial x}(\hat{x}_0) & \dfrac{\partial g}{\partial y}(\hat{x}_0) \end{bmatrix}$$

is called the *Jacobian matrix* at $\hat{x}_0$. The determinant of the Jacobian matrix is called the *Jacobian* of f and g. The Jacobian of f and g is defined by

$$\frac{\partial(f,g)}{\partial(x,y)}=\begin{vmatrix} \dfrac{\partial f}{\partial x} & \dfrac{\partial f}{\partial y} \\ \dfrac{\partial g}{\partial x} & \dfrac{\partial g}{\partial y} \end{vmatrix}=\frac{\partial f}{\partial x}\frac{\partial g}{\partial y}-\frac{\partial f}{\partial y}\frac{\partial g}{\partial x}$$

Definition: Two functions $f:\mathbb{R}\times\mathbb{R}\to\mathbb{R}$ and $g:\mathbb{R}\times\mathbb{R}\to\mathbb{R}$ are said to be

1. *Functionally dependent* if $\dfrac{\partial(f,g)}{\partial(x,y)}=0$
2. *Functionally independent* if $\dfrac{\partial(f,g)}{\partial(x,y)}\neq 0$

Problem-1: Find the Jacobian matrix for the function $F(x,y)=\left(e^{x+y}+y,\, y^2x\right)$.

Solution: Given that $F(x,y)=\left(e^{x+y}+y,\, y^2x\right)$. Let $f(x,y)=e^{x+y}+y$ and $g(x,y)=y^2x$

$$\Rightarrow \frac{\partial f}{\partial x}=e^{x+y}\,,\ \frac{\partial f}{\partial y}=e^{x+y}+1,\ \frac{\partial g}{\partial x}=y^2 \ \text{ and } \ \frac{\partial g}{\partial y}=2xy$$

Then the Jacobian matrix of $F(x,y)=\left(e^{x+y}+y,\, y^2x\right)$ is given by

$$J_F(x,y) = \begin{bmatrix} \frac{\partial f}{\partial x} & \frac{\partial f}{\partial y} \\ \frac{\partial g}{\partial x} & \frac{\partial g}{\partial y} \end{bmatrix}$$

$$= \begin{bmatrix} e^{x+y} & e^{x+y}+1 \\ y^2 & 2xy \end{bmatrix}$$

Problem-2: Show that $u(x,y)=\dfrac{x-y}{1-xy}$ and $v(x,y)=\tan^{-1}x+\tan^{-1}y$ are functionally dependent and hence find a relationship between them.

Solution: Given that $u(x,y)=\dfrac{x-y}{1-xy}$ and $v(x,y)=\tan^{-1}x+\tan^{-1}y$

$$\Rightarrow \frac{\partial u}{\partial x}=\frac{1+y^2}{(1-xy)^2},\ \frac{\partial u}{\partial y}=\frac{1+x^2}{(1-xy)^2},\ \frac{\partial v}{\partial x}=\frac{1}{1+x^2} \text{ and } \frac{\partial v}{\partial y}=\frac{1}{1+y^2}$$

Then $$\frac{\partial(u,v)}{\partial(x,y)} = \begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{vmatrix}$$

$$= \frac{\partial u}{\partial x}\frac{\partial v}{\partial y}-\frac{\partial u}{\partial y}\frac{\partial v}{\partial x}$$

$$=\left(\frac{1+y^2}{(1-xy)^2}\right)\left(\frac{1}{1+y^2}\right)-\left(\frac{1+x^2}{(1-xy)^2}\right)\left(\frac{1}{1+x^2}\right)$$

$$= 0$$

$\Rightarrow\ u(x,y)=\dfrac{x-y}{1-xy}$ and $v(x,y)=\tan^{-1}x+\tan^{-1}y$ are functionally dependent.

Since $v(x,y)=\tan^{-1}x+\tan^{-1}y$, we have $\tan v=\tan\left(\tan^{-1}x+\tan^{-1}y\right)$

$$=\frac{x+y}{1-xy}=u$$

$\Rightarrow\ \tan v = u$ or $v=\tan^{-1}u$

This is a functional relationship between u and v.

Remarks

1. $\dfrac{\partial(u,v)}{\partial(x,y)}\dfrac{\partial(x,y)}{\partial(u,v)}=1$

2. $\dfrac{\partial(u,v)}{\partial(x,y)}=\dfrac{\partial(u,v)}{\partial(r,s)}\dfrac{\partial(r,s)}{\partial(x,y)}$

3. $\dfrac{\partial(u,v,w)}{\partial(x,y,z)}=\begin{vmatrix}\dfrac{\partial u}{\partial x} & \dfrac{\partial u}{\partial y} & \dfrac{\partial u}{\partial z}\\ \dfrac{\partial v}{\partial x} & \dfrac{\partial v}{\partial y} & \dfrac{\partial v}{\partial z}\\ \dfrac{\partial w}{\partial x} & \dfrac{\partial w}{\partial y} & \dfrac{\partial w}{\partial z}\end{vmatrix}$

Exercise

1. Find the Jacobian matrix for the function $F(x,y)=\left(x^2+\cos y,\, ye^x\right)$.
2. Show that $\dfrac{\partial(u,v)}{\partial(x,y)}\dfrac{\partial(x,y)}{\partial(u,v)}=1$.
3. If $x=uv$, $y=\dfrac{u}{v}$ then find $\dfrac{\partial(x,y)}{\partial(u,v)}$.
4. If $u=\dfrac{yz}{x}$, $v=\dfrac{zx}{y}$ and $w=\dfrac{xy}{z}$ then find $\dfrac{\partial(u,v,w)}{\partial(x,y,z)}$.
5. Show that the functions $u=x+y+z$, $v=xy+yz+zx$ and $w=x^2+y^2+z^2$ are functionally dependent and hence find a relation between them
6. Prove that $u=\dfrac{x^2-y^2}{x^2+y^2}$, $v=\dfrac{2xy}{x^2+y^2}$ are functionally dependent and hence find a relation between them.
7. Determine the functional dependence and find the relation between $u=\dfrac{x-y}{x+y}$ and $v=\dfrac{xy}{(x-y)^2}$.

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Maxima & Minima

Definitions:

1. A function $f:\mathbb{R}^n \to \mathbb{R}$ is called a *scalar field*, where $n>1$.
2. A function $f:\mathbb{R}^n \to \mathbb{R}^m$ is called a *vector field* , where $n>1$ and $m>1$.
3. Let $f:\mathbb{R}\times\mathbb{R}\to\mathbb{R}$ and $z=f(x,y)$. A point (x_0,y_0) is said to be a *critical point* or a *stationary point* if $\frac{\partial f}{\partial x}(x_0,y_0)=f_x(x_0,y_0)=0$ and $\frac{\partial f}{\partial y}(x_0,y_0)=f_y(x_0,y_0)=0$.
4. Let $S\subseteq\mathbb{R}\times\mathbb{R}$. A scalar field $f:S\to\mathbb{R}$ is said to have an *absolute maximum* or a *global maximum* at a point (a,b) of S if $|f(x,y)|\le f(a,b)$ for all $(x,y)\in S$. The number $f(a,b)$ is called an absolute maximum value of f on S.
5. Let $S\subseteq\mathbb{R}\times\mathbb{R}$. A scalar field $f:S\to\mathbb{R}$ is said to have an *relative maximum* or a *local maximum* at a point (a,b) of S if there exists a neighborhood B of (a,b) in S such that $|f(x,y)|\le f(a,b)$ for all $(x,y)\in B$. The number $f(a,b)$ is called a local maximum value of f on S.
6. Let $S\subseteq\mathbb{R}\times\mathbb{R}$. A scalar field $f:S\to\mathbb{R}$ is said to have an *absolute minimum* or a *global minimum* at a point (a,b) of S if $|f(x,y)|\ge f(a,b)$ for all $(x,y)\in S$. The number $f(a,b)$ is called an absolute minimum value of f on S.
7. Let $S\subseteq\mathbb{R}\times\mathbb{R}$. A scalar field $f:S\to\mathbb{R}$ is said to have an *relative minimum* or a *local minimum* at a point (a,b) of S if there exists a neighborhood B of (a,b) in S such that $|f(x,y)|\ge f(a,b)$ for all $(x,y)\in B$. The number $f(a,b)$ is called a local minimum value of f on S.

8. Let $S \subseteq \mathbb{R} \times \mathbb{R}$. A point (x_0, y_0) is said to be a *stationary point* of a scalar field $f : S \to \mathbb{R}$ in S if $\frac{\partial f}{\partial x}(x_0, y_0) = f_x(x_0, y_0) = 0$ and $\frac{\partial f}{\partial y}(x_0, y_0) = f_y(x_0, y_0) = 0$. A stationary point $(x_0, y_0) \in S$ is said to be a *saddle point* of $f : S \to \mathbb{R}$ if every neighborhood B of (x_0, y_0) contains points (x, y) such that $f(x, y) < f(x_0, y_0)$ and other points such that $f(x, y) > f(x_0, y_0)$.

Remark: Let $S \subseteq \mathbb{R} \times \mathbb{R}$. The stationary points of a function $f : S \to \mathbb{R}$ are of the following types

a) Points of Maxima
b) Points of Minima
c) Saddle Points

Notation: Let $z = f(x, y)$. We use the following notation.

1. $p = \frac{\partial z}{\partial x} = f_x$
2. $q = \frac{\partial z}{\partial y} = f_y$
3. $r = \frac{\partial^2 z}{\partial x^2} = f_{xx}$
4. $s = \frac{\partial^2 z}{\partial x \partial y} = f_{xy}$
5. $t = \frac{\partial^2 z}{\partial y^2} = f_{yy}$

Here we suppose that both $\frac{\partial z}{\partial x}$ and $\frac{\partial z}{\partial y}$ continuous.

Problem-1: Find the stationary points of $f(x, y) = 2 - (x^2 + y^2)$ for all $(x, y) \in \mathbb{R} \times \mathbb{R}$.

Solution: Given that $f(x,y) = 2 - \left(x^2 + y^2\right)$ for all $(x,y) \in \mathbb{R} \times \mathbb{R}$.

$$\Rightarrow \frac{\partial f}{\partial x} = -2x \quad \text{and} \quad \frac{\partial f}{\partial y} = -2y$$

Clearly , $\frac{\partial f}{\partial x} = 0 \Rightarrow x = 0$

$$\frac{\partial f}{\partial y} = 0 \Rightarrow y = 0$$

Hence $(0,0)$ is a stationary point for the function $f(x,y) = 2 - \left(x^2 + y^2\right)$

and $f(0,0) = 2$.

Since $f(x,y) = 2 - \left(x^2 + y^2\right) \leq 2 = f(0,0)$ for all $(x,y) \in \mathbb{R} \times \mathbb{R}$, it follows that f not only has a local maximum at (0,0) but also an absolute maximum at the origin.

Problem- 2: Find the stationary points of $f(x,y) = x^2 + y^2$ for all $(x,y) \in \mathbb{R} \times \mathbb{R}$.

Solution: Given that $f(x,y) = x^2 + y^2$ for all $(x,y) \in \mathbb{R} \times \mathbb{R}$.

$$\Rightarrow \frac{\partial f}{\partial x} = 2x \quad \text{and} \quad \frac{\partial f}{\partial y} = 2y$$

Clearly, $\frac{\partial f}{\partial x} = 0 \Rightarrow x = 0$

$$\frac{\partial f}{\partial y} = 0 \Rightarrow y = 0$$

Hence $(0,0)$ is a stationary point for the function $f(x,y) = x^2 + y^2$

and $f(0,0) = 0$.

Since $f(x,y) = x^2 + y^2 \geq 0 = f(0,0)$ for all $(x,y) \in \mathbb{R} \times \mathbb{R}$, it follows that f has an absolute minimum at (0,0) .

Problem-3: Find the stationary points of $f(x,y)=xy$ for all $(x,y)\in\mathbb{R}\times\mathbb{R}$.

Solution: Given that $f(x,y)=xy$ for all $(x,y)\in\mathbb{R}\times\mathbb{R}$.

$$\Rightarrow \frac{\partial f}{\partial x}=y \quad \text{and} \quad \frac{\partial f}{\partial y}=x$$

Clearly, $\dfrac{\partial f}{\partial x}=0 \quad \Rightarrow \quad y=0$

$\dfrac{\partial f}{\partial y}=0 \quad \Rightarrow \quad x=0$

Hence $(0,0)$ is a stationary point for the function $f(x,y)=xy$

and $f(0,0)=0$.

Consider a neighborhood centered at the origin and with radius $\delta>0$ in $\mathbb{R}\times\mathbb{R}$. Choose a point $(-a,-b)$ such that $0<\sqrt{a^2+b^2}<\delta$, where a and b are positive real numbers. Then $f(-a,-b)=ab>0=f(0,0)$. Now, choose a point $(u,-v)$ such that $0<\sqrt{u^2+v^2}<\delta$, where u and v are positive real numbers. Then $f(u,v)=-uv<0=f(0,0)$.

$\Rightarrow$ $(0,0)$ is neither point of maximum nor a point of minimum for the given function.

Hence $(0,0)$ is a saddle point for the given function $f(x,y)=xy$.

Taylor's Theorem

Suppose that all the first and second order partial derivatives of $z=f(x,y)$ are continuous on the domain of f. Then

$$f(x,y)=f(x_0,y_0)+(x-x_0)\left[\frac{\partial f}{\partial x}(x_0,y_0)\right]+(y-y_0)\left[\frac{\partial f}{\partial y}(x_0,y_0)\right]+\frac{1}{2}(x-x_0)^2\left[\frac{\partial^2 f}{\partial x^2}(x_0,y_0)\right]+$$

$$+(x-x_0)(y-y_0)\left[\frac{\partial^2 f}{\partial x\partial y}(x_0,y_0)\right]+\frac{1}{2}(y-y_0)^2\left[\frac{\partial^2 f}{\partial y^2}(x_0,y_0)\right]+R_2$$

Where $R_2/\|h\|^2 \to 0$ as $\|h\| \to 0$, where $h = (x - x_0, y - y_0)$.

Test for Maxima and Minima

Let $z = f(x,y)$ have continuous partial derivatives up to second order and suppose that (x_0, y_0) is a critical point of f. Consider the expression the expression

$$\Delta = \left[f_{xx}(x_0,y_0)\right]\left[f_{yy}(x_0,y_0)\right] - \left[f_{xy}(x_0,y_0)\right]^2 \quad \text{or} \quad \Delta = rt - s^2 \quad \text{at } (x_0,y_0).$$

1. If $\Delta > 0$ and $f_{xx}(x_0,y_0) > 0$, then (x_0,y_0) is a local minimum for f.
2. If $\Delta > 0$ and $f_{xx}(x_0,y_0) < 0$, then (x_0,y_0) is a local maximum for f.
3. If $\Delta < 0$, then (x_0,y_0) is a saddle point for f.
4. If $\Delta = 0$ the test fails and further examination of the function is necessary.

Here $f_{xx}(x_0,y_0)$ means that $r = \dfrac{\partial^2 f}{\partial x^2}$ is evaluated at the point (x_0,y_0).

Problem-4: Find the critical points of $f(x,y) = x\sin y$ and determine whether they are local maxima, local minima or saddle points.

Solution: Given that $f(x,y) = x\sin y$ for all $(x,y) \in \mathbb{R} \times \mathbb{R}$

$$\Rightarrow \frac{\partial f}{\partial x} = \sin y \quad \text{and} \quad \Rightarrow \frac{\partial f}{\partial y} = x\cos y$$

Clearly, $\dfrac{\partial f}{\partial x} = 0 \Rightarrow \sin y = 0 \Rightarrow y = n\pi$, where $n \in \mathbb{Z}$

$$\frac{\partial f}{\partial y} = 0 \Rightarrow x\cos y = 0 \Rightarrow x = 0 \text{ because } \cos y \neq 0 \text{ for } y = n\pi$$

Hence the critical points of $f(x,y) = x\sin y$ are $(0, n\pi)$ for $n \in \mathbb{Z}$

Now, $r = \dfrac{\partial^2 f}{\partial x^2} = 0$

$$t = \frac{\partial^2 f}{\partial y^2} = -x \sin y \;\Rightarrow\; f_{yy}(0, n\pi) = 0$$

$$s = \frac{\partial^2 f}{\partial x \partial y} = \cos y \;\Rightarrow\; f_{xy}(0, n\pi) = (-1)^n$$

Then $\Delta = \left[f_{xx}(0, n\pi)\right]\left[f_{yy}(0, n\pi)\right] - \left[f_{xy}(0, n\pi)\right]^2 = -1$

$\Rightarrow \quad \Delta_{(0,n\pi)} < 0$

$\Rightarrow$ all the critical points are saddle points for the give function

$f(x, y) = x \sin y$.

Problem-5: Find the maximum values of $x^3 + y^3 - 3axy$, where $a \neq 0$.

Solution: Given that $f(x, y) = x^3 + y^3 - 3axy$, where $a \neq 0$.

$$\Rightarrow \frac{\partial f}{\partial x} = 3x^2 - 3ay \quad \text{and} \quad \frac{\partial f}{\partial y} = 3y^2 - 3ax$$

Clearly, $\frac{\partial f}{\partial x} = 0 \;\Rightarrow\; 3x^2 - 3ay = 0$

$$\Rightarrow x^2 - ay = 0$$

$$\Rightarrow x^2 = ay \quad \text{or} \quad y = \frac{x^2}{a} \qquad \rightarrow (1)$$

Similarly, $\frac{\partial f}{\partial y} = 0 \;\Rightarrow\; 3y^2 - 3ax = 0$

$$\Rightarrow y^2 - ax = 0 \qquad \rightarrow (2)$$

From (1) and (2), $\frac{x^4}{a^2} - ax = 0$

$\Rightarrow x^4 - a^3 x = 0$

$\Rightarrow x\left(x^3 - a^3\right) = 0$

$\Rightarrow \; x(x-a)(x^2+ax+a^2)=0$

$\Rightarrow \; x=0$ or $x=a$

If $x=0$ then $y=0$

If $x=a$ then $y=a$

Thus, $(0,0)$ and (a,a) are stationary points for the given function

$f(x,y)=x^3+y^3-3axy$.

We have $r=6x$, $t=6y$ and $s=-3a$

At the point $(0,0)$, $\Delta = rt-s^2 = -9a^2<0$

$\Rightarrow$ $(0,0)$ is a saddle point for the given function $f(x,y)=x^3+y^3-3axy$

At the point (a,a), we have $r=6a, t=6a$ and $s=-3a$

$\Rightarrow \Delta_{(a,a)} = rt-s^2 = 36a^2-9a^2 = 27a^2>0$

Case-1

Suppose that $a>0$. Then $r=6a>0$ and $\Delta_{(a,a)}>0$

$\Rightarrow (a,a)$ is a point of minimum for the given function

Case-2

Suppose that $a<0$. Then $r=6a<0$ and $\Delta_{(a,a)}>0$

$\Rightarrow (a,a)$ is a point of maximum for the given function

Constrained Extrema & Lagrange Multipliers

In some problems, we need to maximize or minimize a function $z=f(x,y)$ subject to a condition $g(x,y)=c$, where c is a constant.

1. To find the points of extrema for the given function $z = f(x,y)$ under the constraint $g(x,y) = c$, we use the Lagrange method of multipliers. In this method, we construct the following equations

$$\frac{\partial f}{\partial x} = \lambda \frac{\partial g}{\partial x} \qquad \rightarrow \quad (1)$$

$$\frac{\partial f}{\partial y} = \lambda \frac{\partial g}{\partial y} \qquad \rightarrow \quad (2)$$

$$g(x,y) = c \qquad \rightarrow \quad (3)$$

Here λ is called a Lagrange multiplier. By solving these three equations, we find the values of x, y and λ. After we find all pairs of points (x,y) satisfying these three equations, say $(x_1,y_1),(x_2,y_2),\ldots,(x_n,y_n)$, we evaluate f at these points obtaining a list of numbers $f(x_1,y_1), f(x_2,y_2),\ldots, f(x_n,y_n)$. The absolute maximum will be the largest of these and the absolute minimum the smallest.

2. To find the points of extrema for the given function $w = f(x,y,z)$ subjected to the constraint $g(x,y,z) = c$, we use the Lagrange method of multipliers. In this method, we construct the following equations

$$\frac{\partial f}{\partial x} = \lambda \frac{\partial g}{\partial x} \qquad \rightarrow \quad (1)$$

$$\frac{\partial f}{\partial y} = \lambda \frac{\partial g}{\partial y} \qquad \rightarrow \quad (2)$$

$$\frac{\partial f}{\partial z} = \lambda \frac{\partial g}{\partial z} \qquad \rightarrow \quad (3)$$

$$g(x,y,z) = c \qquad \rightarrow \quad (4)$$

Here λ is called a Lagrange multiplier. By solving these four equations, we find the values of x, y, z and λ. After we find all points (x,y,z) satisfying these four equations, say $(x_1,y_1,z_1),(x_2,y_2,z_2),\ldots,(x_n,y_n,z_n)$, we evaluate f at these points obtaining a list of numbers

$f(x_1, y_1, z_1), f(x_2, y_2, z_1), \ldots, f(x_n, y_n, z_1)$. The absolute maximum will be the largest of these and the absolute minimum the smallest of these values.

Problem-1: Find the extreme values of $f(x, y) = x^2 - y^2$ along the circle C of radius 1 centered at the origin in the xy plane.

Solution: Given that $f(x, y) = x^2 - y^2$. We have to find the extreme values of the given function along the circle $x^2 + y^2 = 1$

Let $g(x, y) = x^2 + y^2 - 1$. Then $\frac{\partial f}{\partial x} = 2x$, $\frac{\partial f}{\partial y} = -2y$, $\frac{\partial g}{\partial x} = 2x$ and $\frac{\partial g}{\partial y} = 2y$

$$\frac{\partial f}{\partial x} = \lambda \frac{\partial g}{\partial x} \quad \Rightarrow \quad 2x = 2\lambda x \qquad \rightarrow \quad (1)$$

$$\frac{\partial f}{\partial y} = \lambda \frac{\partial g}{\partial y} \quad \Rightarrow \quad -2y = 2\lambda y \qquad \rightarrow \quad (2)$$

$$x^2 + y^2 = 1 \qquad \rightarrow \quad (3)$$

Consider $2x = 2\lambda x$

$\Rightarrow \quad 2x(\lambda - 1) = 0$

$\Rightarrow \quad x = 0$ or $\lambda = 1$

Consider $-2y = 2\lambda y$

$\Rightarrow \quad 2y(\lambda + 1) = 0$

$\Rightarrow \quad y = 0$ or $\lambda = -1$

If $x = 0$, $x^2 + y^2 = 1 \quad \Rightarrow \quad y = 1$ *or* -1

If $y = 0$, $x^2 + y^2 = 1 \quad \Rightarrow \quad x = 1$ *or* -1

If $\lambda = 1$ then $y = 0$. In this case, $(1,0)$ and $(-1,0)$ are critical points for the given function $f(x, y) = x^2 - y^2$ along the circle $x^2 + y^2 = 1$

If $\lambda = -1$ then $x = 0$. In this case, $(0,1)$ and $(0,-1)$ are critical points for the given function $f(x,y) = x^2 - y^2$ along the circle $x^2 + y^2 = 1$

Clearly, $f(0,1) = f(0,-1) = -1$ and $f(1,0) = f(-1,0) = 1$. So the maximum and minimum values are 1 and -1.

Problem-2: Find the extreme values of $z = xy$ subject to the condition $x + y = 1$.

Solution: Given that $z = xy$ is subjected to the condition $x + y = 1$

Let $f(x,y) = xy$ and $g(x,y) = x + y - 1$.

Then $\frac{\partial f}{\partial x} = y$, $\frac{\partial f}{\partial y} = x$, $\frac{\partial g}{\partial x} = 1$ and $\frac{\partial g}{\partial y} = 1$

Then $\frac{\partial f}{\partial x} = \lambda \frac{\partial g}{\partial x} \quad \Rightarrow \quad y = \lambda \quad \rightarrow \quad (1)$

$\frac{\partial f}{\partial y} = \lambda \frac{\partial g}{\partial y} \quad \Rightarrow \quad y = \lambda \quad \rightarrow \quad (2)$

$x + y = 1 \quad \rightarrow \quad (3)$

From (1) and (2), we get $x = y = \lambda$

Since $x = y$ and $x + y = 1$, we have $2x = 1 \Rightarrow x = \frac{1}{2}$

Hence $x = y = \frac{1}{2}$

$\Rightarrow \left(\frac{1}{2}, \frac{1}{2}\right)$ is a critical point for the function $f(x,y) = xy$ subjected to the condition $x + y = 1$.

$\Rightarrow f\left(\frac{1}{2}, \frac{1}{2}\right) = \frac{1}{4}$ is a maximum value of $z = xy$ is subjected to the condition $x + y = 1$.

Problem-3: Find the minimum value of $x^2+y^2+z^2$, given that $xyz=a^3$, where $a>0$.

Solution: Let $f(x,y,z)=x^2+y^2+z^2$ and $g(x,y)=xyz-a^3$, where $a>0$

$\Rightarrow \frac{\partial f}{\partial x}=2x$, $\frac{\partial f}{\partial y}=-2y$, $\frac{\partial f}{\partial z}=2z$, $\frac{\partial g}{\partial x}=yz$, $\frac{\partial g}{\partial y}=xz$ and $\frac{\partial g}{\partial z}=xy$

$$\frac{\partial f}{\partial x}=\lambda\frac{\partial g}{\partial x} \quad\Rightarrow\quad 2x=\lambda yz \quad\Rightarrow\quad \frac{x}{yz}=\frac{\lambda}{2} \quad\rightarrow\quad (1)$$

$$\frac{\partial f}{\partial y}=\lambda\frac{\partial g}{\partial y} \quad\Rightarrow\quad 2y=\lambda xz \quad\Rightarrow\quad \frac{y}{xz}=\frac{\lambda}{2} \quad\rightarrow\quad (2)$$

$$\frac{\partial f}{\partial z}=\lambda\frac{\partial g}{\partial z} \quad\Rightarrow\quad 2z=\lambda xy \quad\Rightarrow\quad \frac{z}{xy}=\frac{\lambda}{2} \quad\rightarrow\quad (3)$$

$$xyz=a^3 \quad\rightarrow\quad (4)$$

From equations (1), (2) and (3)

$$\frac{x}{yz}=\frac{y}{xz}=\frac{z}{xy}=\frac{\lambda}{2}$$

From $\frac{x}{yz}=\frac{y}{xz}$, we get $x^2=y^2 \Rightarrow x=y$ or $x=-y$

From $\frac{y}{xz}=\frac{z}{xy}$, we get $y^2=z^2 \Rightarrow y=z$ or $y=-z$

If $x=y=z$ then $xyz=a^3 \Rightarrow x^3=a^3 \Rightarrow x=a$

In this, (a,a,a) is a critical point of $f(x,y,z)=x^2+y^2+z^2$ under the constraint $xyz=a^3$.

If $x=y$ and $y=-z$ then $xyz=a^3 \Rightarrow z^3=a^3 \Rightarrow z=a$.

In this case $(-a,-a,a)$ is a critical point of $f(x,y,z)=x^2+y^2+z^2$ under the constraint $xyz=a^3$.

If $x=-y$ and $y=z$ then $xyz=a^3 \Rightarrow y^3=-a^3 \Rightarrow y=-a$

In this case $(a,-a,-a)$ is a critical point of $f(x,y,z)=x^2+y^2+z^2$ under the constraint $xyz=a^3$.

If $x=-y$ and $y=-z$ then $xyz=a^3 \Rightarrow y^3=a^3 \Rightarrow y=a$

In this case $(-a,a,-a)$ is a critical point of $f(x,y,z)=x^2+y^2+z^2$ under the constraint $xyz=a^3$.

$\Rightarrow f(a,a,a)=3a^2$, $f(-a,-a,a)=3a^2$, $f(a,-a,-a)=3a^2$ and
$f(-a,a,-a)=3a^2$.

Exercise

1. Classify the stationary points of following functions

   a) $z = x^2 + (y-1)^2$ b) $z = x^2 - (y-1)^2$

   c) $z = x^3 - 3xy^2 + y^2$ d) $z = x^3 + y^3 - 3xy$

2. Find the points of maxima, minima and saddle points for $f(x,y) = (3-x)(3-y)(x+y-3)$.

3. Find the maximum and minimum values of $f(x,y) = x^3y^2(1-x-y)$.

4. Find the minimum value of $f(x,y,z) = x^2 + y^2 + z^2$ subjected to $x+y+z=3a$, where $a>0$.

5. Find the maximum and minimum values of $f(x,y,z) = x+y+z$ subjected to $\frac{1}{x}+\frac{1}{y}+\frac{1}{z}=1$.

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