Mean Value Theorems

Rolle's Theorem: If $f:[a,b]\to\mathbb{R}$ is a real - valued function that satisfies the following conditions,

1. f is continuous on $[a,b]$
2. f is differentiable in (a,b)
3. $f(a)=f(b)$

Then there exists a point c with $a<c<b$ such that $f'(c)=0$.

Geometrical Interpretation: Let $y=f(x)$ be the given curve joining the points $(a,f(a))$ and $(b,f(b))$. Suppose that the given curve $y=f(x)$ is a continuous curve without breaks and without any sharp edges. Also, suppose that the given curve is a closed curve that is, $f(a)=f(b)$. Then by the Rolle's theorem, there exists a point $(c,f(c))$ on the curve $y=f(x)$ such that the tangent at this point $(c,f(c))$ is parallel to the $x-$axis.

Problem: Verify Rolle's theorem for the function $f(x) = (x+2)^3(x-3)^4$ in $[2,3]$.

Solution: Given function is $f(x) = (x+2)^3(x-3)^4 \quad \forall\ x\in[-2,3]$

Clearly, this function is both continuous and differentiable on $[-2,3]$.

Also, $f(-2)=f(3)=0$. According to the Rolle's theorem, there exists a point c with $-2<c<3$ such that $f'(c)=0$

Since, $f(x) = (x+2)^3(x-3)^4$, $f'(x)=4(x+2)^3(x-3)^3+3(x+2)^2(x-3)^4$. Then $f'(c)=0$

$\Rightarrow 4(c+2)^3(c-3)^3+3(c+2)^2(c-3)^4=0$

$\Rightarrow 4(c+2)+3(c-3)=0$

$\Rightarrow 7c-1=0$

$\Rightarrow c=\dfrac{1}{7}$

Clearly $c=\dfrac{1}{7}$ belongs to $(-2,3)$ and the Rolle's theorem is verified.

Problem: Find c of the Rolle's theorem for the function $f(x)=x^2-2x-3$ in $[-1,3]$.

Solution: Given function is $f(x)=x^2-2x-3$ for all $x\in[-1,3]$

Clearly, this function is both continuous and differentiable on $[-1,3]$

Also, $f(-1)=f(3)=0$

According to the Rolle's theorem, there exists a point c with $-1<c<3$ such that $f'(c)=0$

Since, $f(x)=x^2-2x-3$, $f'(x) = 2x-2$. Then $f'(c)=0$

$\Rightarrow 2c-2=0$

$\Rightarrow c=1$

Clearly $c=1$ belongs to $(-1,3)$ and the Rolle's theorem is verified.

Lagrange's Theorem: If $f:[a,b]\to\mathbb{R}$ is a real-valued function that satisfies

1. f is continuous on $[a,b]$
2. f is differentiable in (a,b)
3. Then there exists a point c with $a<c<b$ such that

$$f'(c)=\frac{f(b)-f(a)}{b-a}.$$

Geometrical Interpretation: If $y=f(x)$ is a given curve such that it is continuous without any breaks and without any sharp edges. then there exists a point $(c,f(c))$ on the given curve $y=f(x)$ such that the tangent line at this point to the curve $y=f(x)$ is parallel to the straight line joining the points $(a,f(a))$ and $(b,f(b))$. Hence the slope of the tangent line at the point $(c,f(c))$ is equal to the slope of the straight line joining $(a,f(a))$ and $(b,f(b))$. That is $f'(c)=\dfrac{f(b)-f(a)}{b-a}$.

Problem: Verify Lagrange's mean value theorem for $f(x)=x^3-x^2-5x+3$ in $[0,4]$.

Solution: Given function is $f(x) = x^3 - x^2 - 5x + 3$ for all $x \in [0,4]$

Clearly, this function is both continuous and differentiable on $[0,4]$

According to the Lagrange's theorem, there exists a point c with $0 < c < 4$ such that $f'(c) = \dfrac{f(b) - f(a)}{b - a}$

$$\Rightarrow f'(c) = \frac{f(4) - f(0)}{4 - 0} = \frac{31 - 3}{4} = 7$$

$f'(c) = 3c^2 - 2c - 5$ s

Since, $f'(c) = 7$, we get

$$3c^2 - 2c - 5 = 7$$

$$\Rightarrow 3c^2 - 2c - 12 = 0$$

$$\Rightarrow c = \frac{1 \pm \sqrt{37}}{3}$$

Since $c = \dfrac{1 + \sqrt{37}}{3}$ belongs to $(0,4)$, we take $c = \dfrac{1 + \sqrt{37}}{3}$ and hence the Lagrange's theorem verified.

Problem: Show that $|\sin x - \sin y| \le |x - y|$ for all x and y in $\mathbb{R}$.

Solution: Let x and y be any two real numbers with $x < y$. Define $f:[x,y] \to \mathbb{R}$ by $f(t) = \sin t$. Clearly, this function $f(t) = \sin t$ is both continuous and differentiable on $[x,y]$ and $f'(t) = \cos t$ for all $t \in [x,y]$

By the Lagrange's mean value theorem, there exists a point c with $x < c < y$

such that $f'(c) = \dfrac{f(y) - f(x)}{y - x}$

$$\Rightarrow \cos c = \frac{\sin y - \sin x}{y - x}$$

$$\Rightarrow |\cos c| = \left| \frac{\sin y - \sin x}{y - x} \right|$$

$$\Rightarrow |\sin y - \sin x| = |\cos c|\,|y - x| \quad \text{or} \quad |\sin x - \sin y| = |\cos c|\,|x - y|$$

Since $|\cos c| \leq 1$, we have $|\sin x - \sin y| = |\cos c|\,|x-y| \leq |x-y|$

$\Rightarrow |\sin x - \sin y| \leq |x-y|$

Problem: If $a<b$ then show that $\dfrac{b-a}{1+b^2} \leq \tan^{-1} b - \tan^{-1} a \leq \dfrac{b-a}{1+a^2}$. Deduce

$\dfrac{\pi}{4} + \dfrac{3}{25} \leq \tan^{-1}\dfrac{4}{3} \leq \dfrac{\pi}{4} + \dfrac{1}{6}$.

Solution: Suppose that a and b are any two real numbers with $a<b$.
Define $f:[a,b] \to \mathbb{R}$ by $f(x) = \tan^{-1} x$ for all $x \in [a,b]$.

Clearly $f(x) = \tan^{-1} x$ is continuous and differentiable on $[a,b]$ and $f'(x) = \dfrac{1}{1+x^2}$

By the Lagrange's mean values theorem, there exists a point c with $a<c<b$ such that $f'(c) = \dfrac{f(b)-f(a)}{b-a}$

$$\frac{1}{1+c^2} = \frac{\tan^{-1} b - \tan^{-1} a}{b-a} \qquad \rightarrow \qquad (1)$$

Since $a<c<b$, we have $a^2 \leq c^2 \leq b^2$

$\Rightarrow 1+a^2 \leq 1+c^2 \leq 1+b^2$

$$\Rightarrow \frac{1}{1+b^2} \leq \frac{1}{1+c^2} \leq \frac{1}{1+a^2} \qquad \rightarrow \quad (2)$$

From (1) and (2), we get

$$\frac{1}{1+b^2} \leq \frac{\tan^{-1} b - \tan^{-1} a}{b-a} \leq \frac{1}{1+a^2}$$

$$\Rightarrow \frac{b-a}{1+b^2} \leq \tan^{-1} b - \tan^{-1} a \leq \frac{b-a}{1+a^2}.$$

Taking $b = \dfrac{4}{3}$ and $a=1$, we get

$$\Rightarrow \frac{3}{25} \leq \tan^{-1}\frac{4}{3} - \tan^{-1} 1 \leq \frac{1}{6}$$

$$\Rightarrow \frac{3}{25} \le \tan^{-1}\frac{4}{3} - \frac{\pi}{4} \le \frac{1}{6}$$

$$\Rightarrow \frac{\pi}{4} + \frac{3}{25} \le \tan^{-1}\frac{4}{3} \le \frac{\pi}{4} + \frac{1}{6}.$$

Problem: Show that $x < \sin^{-1} x < \frac{x}{\sqrt{1-x^2}}$ for $0 < x < 1$.

Solution: Let x be a real number with $0 < x < 1$. Define $f : [0, x] \to \mathbb{R}$ by $f(t) = \sin^{-1} t$

Clearly $f(t) = \sin^{-1} t$ is both continuous and differentiable on $[0, x]$. Then by Lagrange's mean value theorem there exists a point c with $0 < c < x$ such that $f'(c) = \frac{f(x) - f(0)}{x - 0} = \frac{\sin^{-1} x}{x}$

Since $f(t) = \sin^{-1} t$, we have $f'(t) = \frac{1}{\sqrt{1-t^2}}$ for all $t \in [0, x]$

Hence $f'(c) = \frac{\sin^{-1} x}{x} \Rightarrow \frac{1}{\sqrt{1-c^2}} = \frac{\sin^{-1} x}{x} \qquad \to (1)$

Since $0 < c < x$, $0 < c^2 < x^2$

$$\Rightarrow \quad 0 > -c^2 > -x^2$$

$$\Rightarrow \quad 1 > 1 - c^2 > 1 - x^2$$

$$\Rightarrow \quad 1 < \frac{1}{\sqrt{1-c^2}} < \frac{1}{\sqrt{1-x^2}} \qquad \to \qquad (2)$$

From (1) and (2), $\quad 1 < \frac{\sin^{-1} x}{x} < \frac{1}{\sqrt{1-x^2}}$

$$\Rightarrow \quad x < \sin^{-1} x < \frac{x}{\sqrt{1-x^2}} \quad \text{for} \quad 0 < x < 1.$$

Problem: For $x > 0$, show that $1 + x < e^x < 1 + xe^x$.

Solution: Let x be any positive real number. Define $f:[0,x]\to\mathbb{R}$ by $f(t)=e^t$. Clearly $f(t)=e^t$ is continuous and differentiable on $[0,x]$. Also, $f'(t)=e^t$ on $[0,x]$. Then by Lagrange's mean value theorem there exists a point c with $0<c<x$ such that $f'(c)=\dfrac{f(x)-f(0)}{x-0}=\dfrac{e^x-1}{x}$

$$\Rightarrow\ e^c=\frac{e^x-1}{x}\qquad\to\ (1)$$

Hence $0<c<x$, $e^0<e^c<e^x$

$$\Rightarrow\ 1<e^c<e^x$$

$$\Rightarrow\ 1<\frac{e^x-1}{x}<e^x$$

$$\Rightarrow\ x<e^x-1<xe^x$$

$$\Rightarrow\ 1+x<e^x<1+xe^x\quad \text{For } x>0.$$

Problem: Prove that $\dfrac{\pi}{6}+\dfrac{1}{5\sqrt{3}}<\sin^{-1}\left(\dfrac{3}{5}\right)<\dfrac{\pi}{6}+\dfrac{1}{8}$.

Solution: Define $f:[a,b]\to\mathbb{R}$ such that $f(x)=\sin^{-1}x$.

Clearly $f(x)=\sin^{-1}x$ is both continuous and differentiable on $[a,b]$ and $f'(x)=\dfrac{1}{\sqrt{1-x^2}}$. Then by Lagrange's mean value theorem there exists a point c with $a<c<b$ such that $f'(c)=\dfrac{f(b)-f(a)}{b-a}=\dfrac{\sin^{-1}b-\sin^{-1}a}{b-a}$

$$\Rightarrow\ \frac{1}{\sqrt{1-c^2}}=\frac{\sin^{-1}b-\sin^{-1}a}{b-a}\qquad\to\ (1)$$

Since $a<c<b$, we have $a^2<c^2<b^2$

$$\Rightarrow\ -a^2>-c^2>-b^2$$

$$\Rightarrow\ 1-a^2>1-c^2>1-b^2$$

$\Rightarrow \sqrt{1-a^2} > \sqrt{1-c^2} > \sqrt{1-b^2}$

$$\Rightarrow \frac{1}{\sqrt{1-a^2}} < \frac{1}{\sqrt{1-c^2}} < \frac{1}{\sqrt{1-b^2}} \qquad \rightarrow (2)$$

From (1) and (2),

$$\frac{1}{\sqrt{1-a^2}} < \frac{\sin^{-1} b - \sin^{-1} a}{b-a} < \frac{1}{\sqrt{1-b^2}}$$

$$\Rightarrow \frac{b-a}{\sqrt{1-a^2}} < \sin^{-1} b - \sin^{-1} a < \frac{b-a}{\sqrt{1-b^2}}$$

Put $a = 1/2$ and $b = 3/5$. Then we get

$$\frac{\pi}{6} + \frac{1}{5\sqrt{3}} < \sin^{-1}\left(\frac{3}{5}\right) < \frac{\pi}{6} + \frac{1}{8}.$$

Problem: Prove that $\frac{\pi}{3} - \frac{1}{5\sqrt{3}} > \cos^{-1}\left(\frac{3}{5}\right) > \frac{\pi}{3} - \frac{1}{8}$.

Solution: Define $f:[a,b] \rightarrow \mathbb{R}$ such that $f(x) = \cos^{-1} x$.

Clearly $f(x) = \cos^{-1} x$ is both continuous and differentiable on $[a,b]$ and $f'(x) = \frac{-1}{\sqrt{1-x^2}}$. Then by Lagrange's mean value theorem there exists a point c with $a < c < b$ such that $f'(c) = \frac{f(b)-f(a)}{b-a} = \frac{\cos^{-1} b - \cos^{-1} a}{b-a}$

$$\Rightarrow \frac{-1}{\sqrt{1-c^2}} = \frac{\cos^{-1} b - \cos^{-1} a}{b-a} \qquad \rightarrow (1)$$

Since $a < c < b$, we have $a^2 < c^2 < b^2$

$\Rightarrow -a^2 > -c^2 > -b^2$

$\Rightarrow 1-a^2 > 1-c^2 > 1-b^2$

$\Rightarrow \sqrt{1-a^2} > \sqrt{1-c^2} > \sqrt{1-b^2}$

$$\Rightarrow \frac{-1}{\sqrt{1-a^2}} > \frac{-1}{\sqrt{1-c^2}} > \frac{-1}{\sqrt{1-b^2}} \qquad \rightarrow (2)$$

From (1) and (2),

$$\frac{-1}{\sqrt{1-a^2}} > \frac{\cos^{-1} b - \cos^{-1} a}{b-a} > \frac{-1}{\sqrt{1-b^2}}$$

$$\Rightarrow \frac{-(b-a)}{\sqrt{1-a^2}} > \cos^{-1} b - \cos^{-1} a > \frac{-(b-a)}{\sqrt{1-b^2}}$$

$$\Rightarrow \frac{(b-a)}{\sqrt{1-a^2}} < \cos^{-1} a - \cos^{-1} b < \frac{b-a}{\sqrt{1-b^2}}$$

Put $a = 1/2$ and $b = 3/5$. Then we get

$$\frac{\pi}{3} - \frac{1}{5\sqrt{3}} > \cos^{-1}\left(\frac{3}{5}\right) > \frac{\pi}{3} - \frac{1}{8}.$$

Problem: Prove that $\tan x > x$ where $0 < x < \pi/2$.

Solution: Suppose that $0 < x < \pi/2$. choose a real number a such that $0 < a < x < \pi/2$. Define $f : [a, x] \to \mathbb{R}$ by $f(t) = \tan t$. Clearly $f(t) = \tan t$ is both continuous and differentiable on $[a, x]$. Also, $f'(t) = \sec^2 t$. Then by Lagrange's mean value theorem there exists a point c with $0 < c < x$ such that $f'(c) = \frac{f(x) - f(a)}{x-a} = \frac{\tan x - \tan a}{x-a}$

$$\Rightarrow \sec^2 c = \frac{\tan x - \tan a}{x-a}$$

$$\Rightarrow \sec^2 c = \lim_{a \to 0} \frac{\tan x - \tan a}{x-a} = \frac{\tan x}{x}$$

$$\Rightarrow \frac{\tan x}{x} = \sec^2 c > 1$$

$$\Rightarrow \tan x > x \quad \text{for} \quad 0 < x < \pi/2.$$

Problem: Using Lagrange's mean value theorem, show that if $f'(x)=0$ then $f(x)$ is a constant

Solution: Suppose that $f'(x)=0\ \forall x$

Let a and b be any two real numbers with $a<b$. Since f is differentiable on $\mathbb{R}$, the function f is differentiable and continuous on $[a,b]$. By the Lagrange's mean value theorem, there exists a real number c with $a<c<b$ such that $f'(c)=\dfrac{f(b)-f(a)}{b-a}$

$$\Rightarrow \quad \frac{f(b)-f(a)}{b-a}=0$$

$$\Rightarrow \quad f(b)-f(a)=0$$

$$\Rightarrow \quad f(b)=f(a)$$

Thus, $f(b)=f(a)$ for all a and b

$\Rightarrow \quad f$ is a constant function

Problem: Using Lagrange's mean value theorem, show that if $f'(x)>0$ then $f(x)$ is monotonically increasing

Solution: Suppose that $f'(x)>0\ \forall x$

Let a and b be any two real numbers with $a<b$. Since f is differentiable on $\mathbb{R}$, the function f is differentiable and continuous on $[a,b]$. By the lagrange's mean value theorem, there exists a real number c with $a<c<b$ such that $f'(c)=\dfrac{f(b)-f(a)}{b-a}$

$$\Rightarrow \quad \frac{f(b)-f(a)}{b-a}>0$$

$$\Rightarrow \quad f(b)-f(a)>0$$

$$\Rightarrow \quad f(b)>f(a)$$

Thus, $f(b) > f(a)$ for all a and b

$\Rightarrow$ f is monotonically increasing function

Problem: Using Lagrange's mean value theorem, show that if $f'(x) < 0$ then $f(x)$ is monotonically decreasing

Solution: Suppose that $f'(x) < 0 \ \forall x$

Let a and b be any two real numbers with $a < b$. Since f is differentiable on $\mathbb{R}$, the function f is differentiable and continuous on $[a,b]$. By the Lagrange's mean value theorem, there exists a real number c with $a < c < b$ such that $f'(c) = \dfrac{f(b)-f(a)}{b-a}$

$$\Rightarrow \frac{f(b)-f(a)}{b-a} < 0$$

$$\Rightarrow f(b)-f(a) < 0$$

$$\Rightarrow f(b) < f(a)$$

Thus, $f(b) < f(a)$ for all a and b

$\Rightarrow$ f is monotonically decreasing function

Cauchy's Theorem: If $f:[a,b] \to \mathbb{R}$ and $g:[a,b] \to \mathbb{R}$ are any two real-valued functions satisfying

1. f and g are continuous on $[a,b]$
2. f and g differentiable in (a,b)
3. $g'(x) \neq 0 \ \forall x \in (a,b)$

Then there exists a point c with $a < c < b$ such that $\dfrac{f'(c)}{g'(c)} = \dfrac{f(b)-f(a)}{g(b)-g(a)}$

Problem: Prove Cauchy's mean value theorem by using Rolle's theorem

Solution: Suppose that $f:[a,b]\to\mathbb{R}$ and $g:[a,b]\to\mathbb{R}$ are any two real-valued functions satisfying

1. f and g are continuous on $[a,b]$
2. f and g differentiable in (a,b)
3. $g'(x)\neq 0\ \forall x\in(a,b)$

Define $F:[a,b]\to\mathbb{R}$ by $F(x)=f(x)\big(g(b)-g(a)\big)-g(x)\big(f(b)-f(a)\big)$

Since f and g are continuous on $[a,b]$ and differentiable in (a,b), the function F is also continuous on $[a,b]$ and differentiable in (a,b).

Also, $F(a)=f(a)g(b)-g(a)f(b)$ & $F(b)=f(a)g(b)-g(a)f(b)$

$\Rightarrow F(a)=F(b)$

By the Rolle's theorem, there exists a real number c with $a<c<b$ such that

$F'(c)=0 \qquad \to (1)$

But $F'(c)=f'(c)\big(g(b)-g(a)\big)-g'(c)\big(f(b)-f(a)\big) \qquad \to (2)$

From (1) and (2), we get $f'(c)\big(g(b)-g(a)\big)-g'(c)\big(f(b)-f(a)\big)=0$

$\Rightarrow f'(c)\big(g(b)-g(a)\big)=g'(c)\big(f(b)-f(a)\big)$

$$\Rightarrow \frac{f'(c)}{g'(c)}=\frac{f(b)-f(a)}{g(b)-g(a)}$$

Problem: Find c of Cauchy's mean value theorem for $f(x)=\sqrt{x}$ and $g(x)=\dfrac{1}{\sqrt{x}}$ in $[a,b]$ where $0<a<b$.

Solution: Let a and b any two real numbers such that $0<a<b$. Define $f:[a,b]\to\mathbb{R}$ and $g:[a,b]\to\mathbb{R}$ by $f(x)=\sqrt{x}$ and $g(x)=\dfrac{1}{\sqrt{x}}$

Clearly f and g are continuous on $[a,b]$. f and g differentiable in (a,b)

also, $f'(x)=\dfrac{1}{2\sqrt{x}}$ and $g(x)=\dfrac{-1}{2x^{3/2}}$

By the Cauchy's mean value theorem, there exists a real number c with $a<c<b$ such that $\dfrac{f'(c)}{g'(c)}=\dfrac{f(b)-f(a)}{g(b)-g(a)}$

$$\Rightarrow\ -c=\frac{\sqrt{b}-\sqrt{a}}{\frac{1}{\sqrt{b}}-\frac{1}{\sqrt{a}}}$$

$$\Rightarrow\ -c=-\sqrt{ab}$$

$$\Rightarrow\ c=\sqrt{ab}$$

Taylor's Theorem: If $f:[a,b]\to\mathbb{R}$ satisfies the following conditions

1. $f, f', f'', \dots$ and $f^{(n-1)}$ exist and are continuous on $[a,b]$
2. $f^{(n)}$ exists in (a,b)

Then there exists a real number c with $a<c<b$ such that

$$f(b)=f(a)+(b-a)\,f'(a)+\frac{(b-a)^2}{2!}\,f''(a)+\dots+\frac{(b-a)^{n-1}}{(n-1)!}\,f^{(n-1)}(a)+R_n$$

Where $R_n=\dfrac{(b-a)^p\,(b-c)^{n-p}\,f^{(n)}(c)}{p\,(n-1)!}$ for $p\in\mathbb{N}$

Remarks:

1. $e^x=1+x+\dfrac{x^2}{2!}+\dfrac{x^3}{3!}+\dots$

2. $\sin x=x-\dfrac{x^3}{3!}+\dfrac{x^5}{5!}-\dfrac{x^7}{7!}+\dots$

3. $\cos x=1-\dfrac{x^2}{2!}+\dfrac{x^4}{4!}-\dfrac{x^6}{6!}+\dots$

Problem: Write Taylor's series expansion of $\sin 2x$ about $x = \frac{\pi}{2}$.

Solution: Let $f(t) = \sin 2t$

$\Rightarrow$ $f'(t) = 2\cos 2t$, $f''(t) = -4\sin 2t$, $f'''(t) = -8\cos 2t$, $f^{(4)}(t) = 16\sin 2t, \ldots$

$\Rightarrow$ $f(t) = \sin 2t$ is differentiable and $f', f'', f''', \ldots$ exist and are all continuous on .

By the Taylor's theorem,

$$f(b) = f(a) + (b-a)\, f'(a) + \frac{(b-a)^2}{2!}\, f''(a) + \ldots + \frac{(b-a)^{n-1}}{(n-1)!}\, f^{(n-1)}(a)$$

Take $b = x$ *and* $a = \frac{\pi}{2}$. then we get

$$\sin 2x = \sin\pi + 2\left(x - \frac{\pi}{2}\right)\cos\pi - \frac{\left(x-\frac{\pi}{2}\right)^2}{2!}\sin\pi - 8\,\frac{\left(x-\frac{\pi}{2}\right)^3}{3!}\cos\pi + 16\,\frac{\left(x-\frac{\pi}{2}\right)^4}{4!}\sin\pi - \ldots$$

$$= -2\left(x - \frac{\pi}{2}\right) + 8\,\frac{\left(x-\frac{\pi}{2}\right)^3}{3!} + \ldots$$

$$\Rightarrow \quad \sin 2x = \; = -2\left(x - \frac{\pi}{2}\right) + \frac{4}{3}\left(x - \frac{\pi}{2}\right)^3 + \ldots$$

Exercise

1. Verify Rolle's theorem for $f(x) = 2x^3 + x^2 + -4x - 2$ in $\left[-\sqrt{3}, \sqrt{3}\right]$.
2. Verify Rolle's theorem for $f(x) = e^x \sin x$ in $[0, \pi]$.
3. Verify Rolle's theorem for $f(x) = x\,(x+3)\,e^{-x/2}$ in $[-3, 0]$.
4. Verify Lagrange's mean value theorem for $f(x) = \cos x$ in $\left[0, \frac{\pi}{2}\right]$.
5. Verify Lagrange's mean value theorem for $f(x) = \log_e^x$ in $[1, e]$.

6. Verify Lagrange's mean value theorem for $f(x) = x(x-1)(x-2)$ in $\left[0, \frac{1}{2}\right]$.

7. Verify Cauchy's mean value theorem for $f(x) = x^2$, $g(x) = x^3$ in $[1,2]$.

8. Verify Cauchy's mean value theorem for $f(x) = \sin x$, $g(x) = \cos x$ in $\left[0, \frac{\pi}{2}\right]$.

9. Verify Cauchy's mean value theorem for $f(x) = e^x$, $g(x) = e^{-x}$ in $[3,7]$

10. Write Taylor's series $f(x) = (1-x)^{5/2}$ with Lagrange's form of remainder Upto 3 terms in the interval $[0,1]$

11. Obtain the Taylor's series expansion of e^x about $x = -1$.

12. Expand $f(x) = \log \sin x$ about $x = 3$ in Taylor's series.

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