Eigen values & Eigen vectors

Definition: Let $A=\left[a_{ij}\right]_{n\times n}$ be any real or complex square matrix. Consider the matrix equation $AX=\lambda X$ or $(A-\lambda I)X=O$. A value of λ for which $AX=\lambda X$ has a solution $X\neq 0$ is called an *eigen value* or *characteristic value* of the matrix A. The corresponding solution $X\neq 0$ to the homogeneous system $(A-\lambda I)X=O$ of linear equations is called an *eigen vector* or *characteristic vector* of the matrix A.

Remarks: Let $A=\left[a_{ij}\right]_{n\times n}$ be any real or complex square matrix. If $\det(A-\lambda I)\neq 0$, then the homogeneous system $(A-\lambda I)X=O$ of linear equations has a unique solution $X=O$. The homogeneous system $(A-\lambda I)X=O$ has infinitely many non-trivial solutions provided $\det(A-\lambda I)=0$

Definition: Let $A=\left[a_{ij}\right]_{n\times n}$ be any real or complex square matrix. Consider the system $AX=\lambda X$ or $(A-\lambda I)X=O$. The equation $\det(A-\lambda I)=0$ is called a *characteristic equation* of the matrix A.

The characteristic polynomial $\det(A-\lambda I)$ is an $n\,th$ degree polynomial in λ and its roots are called eigen values of the matrix A. The set of all eigen values of the matrix A is the *spectrum* of A. The largest of the absolute values of the eigen values of A is called the *spectral radius* of A.

Definition: Two square matrices A and B are said to be *similar matrices* if there exists a non-singular matrix P such that $B=P^{-1}AP$.

Problem: Find the eigen values and corresponding eigen vectors of

$$A=\begin{bmatrix}-5 & 2\\ 2 & -2\end{bmatrix}$$

Solution: Given $A=\begin{bmatrix}-5 & 2\\ 2 & -2\end{bmatrix}$. Then $A-\lambda I=\begin{bmatrix}-5-\lambda & 2\\ 2 & -2-\lambda\end{bmatrix}$

The characteristic polynomial of A is $\det(A-\lambda I)=0$

$\Rightarrow (-5-\lambda)(-2-\lambda)-4=0$

$\Rightarrow (5+\lambda)(2+\lambda)+4=0$

$\Rightarrow \lambda^2+7\lambda+6=0$

$\Rightarrow (\lambda+1)(\lambda+6)=0$

$\Rightarrow \lambda=-1$ *or* -6. Put $\lambda_1=-1$ and $\lambda_2=-6$

Then $\lambda_1=-1$ and $\lambda_2=-6$ are eigen values of the given matrix A.

First we find eigen vectors corresponding to $\lambda_1=-1$. For this, consider $(A-\lambda I)X=O$. Put $\lambda=\lambda_1=-1$

$\Rightarrow (A+I)X=O$

$\Rightarrow \begin{bmatrix}-4 & 2\\ 2 & -1\end{bmatrix}\begin{bmatrix}x_1\\ x_2\end{bmatrix}=\begin{bmatrix}0\\ 0\end{bmatrix}$, where $x=\begin{bmatrix}x_1\\ x_2\end{bmatrix}$

Now $(A+I)=\begin{bmatrix}-4 & 2\\ 2 & -1\end{bmatrix}$

Apply $R_2\to 2R_2+R_1$

$$(A+I) \sim \begin{bmatrix}-4 & 2\\ 0 & 0\end{bmatrix}$$

Take $x_2=k$, where k is a constant

From the first row of $(A+I)\sim\begin{bmatrix}-4 & 2\\ 0 & 0\end{bmatrix}$, we have $-4x_1+2x_2=0$

$-2x_1+x_2=0 \Rightarrow x_1=\dfrac{k}{2}$

$X=\begin{bmatrix}x_1\\ x_2\end{bmatrix}=\begin{bmatrix}\frac{k}{2}\\ k\end{bmatrix}$ is a general solution of $(A+I)X=O$

$\Rightarrow X_1=\begin{bmatrix}1\\ 2\end{bmatrix}$, $X_2=\begin{bmatrix}-1\\ -2\end{bmatrix}$, $X_3=\begin{bmatrix}2\\ 4\end{bmatrix}$, ... are eigen vectors corresponding to $\lambda_1=-1$

Now we find eigen vectors corresponding to $\lambda_2=-6$

Consider $(A-\lambda I)X=O$ and put $\lambda=\lambda_2=-6$

$\Rightarrow \; (A+6I)X = O$

$\Rightarrow \begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$, where $x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$

Take $(A+I) = \begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix}$

Apply $R_2 \to R_2 - 2R_1$

$$(A+6\,I) \sim \begin{bmatrix} 1 & 2 \\ 0 & 0 \end{bmatrix}$$

Choose $x_2 = k$, where k is a constant

From the first row of $(A+6\,I) \sim \begin{bmatrix} 1 & 2 \\ 0 & 0 \end{bmatrix}$, we have $x_1 + 2x_2 = 0$

$\Rightarrow \; x_1 = -2k$

$\Rightarrow X = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} -2k \\ k \end{bmatrix}$ is a general solution of $(A+6I)X = O$

$\Rightarrow X_1 = \begin{bmatrix} -2 \\ 1 \end{bmatrix}$, $X_2 = \begin{bmatrix} 2 \\ -1 \end{bmatrix}$, $X_3 = \begin{bmatrix} -4 \\ 2 \end{bmatrix}, \ldots$ are eigen vectors corresponding to $\lambda_2 = -6$.

Problem: Find the eigen values and corresponding eigen vectors of $A = \begin{bmatrix} 2 & 2 & 1 \\ 1 & 3 & 1 \\ 1 & 2 & 2 \end{bmatrix}$.

Solution: Given $A = \begin{bmatrix} 2 & 2 & 1 \\ 1 & 3 & 1 \\ 1 & 2 & 2 \end{bmatrix}$. Then $A - \lambda I = \begin{bmatrix} 2-\lambda & 2 & 1 \\ 1 & 3-\lambda & 1 \\ 1 & 2 & 2-\lambda \end{bmatrix}$

The characteristic equation of A is $\det(A - \lambda I) = 0$.

$$\Rightarrow \det \begin{bmatrix} 2-\lambda & 2 & 1 \\ 1 & 3-\lambda & 1 \\ 1 & 2 & 2-\lambda \end{bmatrix} = 0$$

$$\Rightarrow (2-\lambda)\left[(3-\lambda)(2-\lambda)-2\right] - 2\left[(2-\lambda)-1\right] + \left[2-(3-\lambda)\right] = 0$$

$$\Rightarrow \; \lambda^3 - 7\lambda^2 + 11\lambda - 5 = 0$$

$\Rightarrow \ (\lambda - 5)(\lambda - 1)^2 = 0$

$\Rightarrow \ \lambda = 1, 1 \ \ or \ \ 5$

$\Rightarrow \lambda = 1 \ \ or \ \ 5$ are eigen values of A. Put $\lambda_1 = 1$ and $\lambda_2 = 5$

First we find eigen vectors corresponding to $\lambda_1 = 1$. For this, consider $(A - \lambda I)X = O$. Put $\lambda = \lambda_1 = 1$

$\Rightarrow (A - I)X = O$

$$\Rightarrow \begin{bmatrix} 1 & 2 & 1 \\ 1 & 2 & 1 \\ 1 & 2 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}, \text{ where } x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

Consider $A - I = \begin{bmatrix} 1 & 2 & 1 \\ 1 & 2 & 1 \\ 1 & 2 & 1 \end{bmatrix}$

Apply $R_2 \to R_2 - R_1 \ \ and \ \ R_3 \to R_3 - R_1$

$$A - I \sim \begin{bmatrix} 1 & 2 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

Choose $x_3 = k_1$ and $x_2 = k_2$, where $k_1 \ \ and \ \ k_2$ are constants

From the first row of $A - I \sim \begin{bmatrix} 1 & 2 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$, we have $x_1 + 2x_2 + x_3 = 0$

$\Rightarrow \ x_1 = -(k_1 + 2k_2)$

$$\Rightarrow X = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -(k_1 + 2k_2) \\ k_2 \\ k_1 \end{bmatrix}$$ is a general solution of $(A - I)X = O$

$\Rightarrow X_1 = \begin{bmatrix} -3 \\ 1 \\ 1 \end{bmatrix}, \quad X_2 = \begin{bmatrix} 3 \\ -1 \\ -1 \end{bmatrix}, \quad X_3 = \begin{bmatrix} -1 \\ -1 \\ 1 \end{bmatrix}, \ldots$ are eigen vectors corresponding to $\lambda_1 = 1$.

Now we find eigen vectors corresponding to $\lambda_2 = 5$

Consider $(A - \lambda I)X = O$ and put $\lambda = \lambda_2 = 5$

$\Rightarrow (A - 5I)X = O$

$$\Rightarrow \begin{bmatrix} -3 & 2 & 1 \\ 1 & -2 & 1 \\ 1 & 2 & -3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}, \quad \text{where} \quad x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

Consider $A - 5I = \begin{bmatrix} -3 & 2 & 1 \\ 1 & -2 & 1 \\ 1 & 2 & -3 \end{bmatrix}$

Apply $R_2 \to 3R_2 + R_1$ and $R_3 \to 3R_3 + R_1$

$$A - 5I \sim \begin{bmatrix} -3 & 2 & 1 \\ 0 & -4 & 4 \\ 0 & 8 & -8 \end{bmatrix}$$

Apply $R_3 \to R_3 + 2R_2$

$$A - 5I \sim \begin{bmatrix} -3 & 2 & 1 \\ 0 & -4 & 4 \\ 0 & 0 & 0 \end{bmatrix}$$

Apply $R_2 \to \dfrac{R_2}{4}$

$$A - 5I \sim \begin{bmatrix} -3 & 2 & 1 \\ 0 & -1 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

From the second row, we have $-x_2 + x_3 = 0 \Rightarrow x_3 = x_2$

Choose $x_3 = k$, where k is a constant

$\Rightarrow x_2 = k$ and from the first row we have $-3x_1 + 2x_2 + x_3 = 0 \Rightarrow -3x_1 + 3k = 0 \Rightarrow x_1 = k$

$$\Rightarrow X = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} k_1 \\ k_1 \\ k_1 \end{bmatrix} \text{ is a general solution of } (A - 5I)X = O$$

$$\Rightarrow X_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix},\quad X_2 = \begin{bmatrix} 2 \\ 2 \\ 2 \end{bmatrix}, \ldots$$ are eigen vectors corresponding to $\lambda_2 = 5$.

Exercise

1. Find eigen values and corresponding eigen vectors of the following real matrices

a. $\begin{bmatrix} 3 & -5 & -4 \\ -5 & -6 & -5 \\ -4 & -5 & 3 \end{bmatrix}$ b. $\begin{bmatrix} 5 & 5 & -4 \\ 4 & 5 & -4 \\ -1 & -1 & 2 \end{bmatrix}$ c. $\begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 1 & 1 & 2 \end{bmatrix}$ d. $\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$

e. $\begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 1 & 2 & 0 \end{bmatrix}$ f. $\begin{bmatrix} 8 & -3 & -6 & -6 \\ 4 & -1 & -4 & -4 \\ 3 & -5 & 1 & 1 \\ 0 & 2 & -4 & -4 \end{bmatrix}$ g. $\begin{bmatrix} 2 & 3 & 11 \\ 0 & 3 & 17 \\ 0 & 0 & -2 \end{bmatrix}$ h. $\begin{bmatrix} 3 & 2 & 2 \\ 1 & 2 & 2 \\ -1 & -1 & 0 \end{bmatrix}$

i. $\begin{bmatrix} 2 & 2 & 0 \\ 2 & 5 & 0 \\ 0 & 0 & 3 \end{bmatrix}$

2. Find eigen values and corresponding eigen vectors of the following complex matrices

a. $\begin{bmatrix} 4 & 1-3i \\ 1+3i & 7 \end{bmatrix}$ b. $\begin{bmatrix} 3i & 2+i \\ -2+i & -i \end{bmatrix}$ c. $\begin{bmatrix} 2 & 3+i \\ 3-4i & 2 \end{bmatrix}$ d. $\begin{bmatrix} 3i & 2+i \\ -2+i & -i \end{bmatrix}$

d. $\begin{bmatrix} 2i & 3i \\ 3i & -i \end{bmatrix}$

Properties of Eigen Values

Theorem-1: If λ is an eigen value of a square matrix A then $\lambda + k$ is an eigen value of $A + kI$.

Proof: Let λ be an eigen value of a square matrix A. Suppose that X is an eigen vector corresponding to λ of A and k is a constant.

Since X is an eigen vector corresponding to λ of A, $AX = \lambda X$

Now $(A + kI)X = AX + kIX = \lambda X + kX = (\lambda + k)X$

$\Rightarrow (A + kI)X = (\lambda + k)X$

$\Rightarrow \lambda + k$ is an eigen value of $A + kI$.

Theorem-2: If λ is an eigen value of a square matrix A then $k\lambda$ is an eigen value of kA.

Proof: Let λ be an eigen value of a square matrix A. Suppose that X is an eigen vector corresponding to λ of A and k is a constant.

Since X is an eigen vector corresponding to λ of A, $AX = \lambda X$

Now $(kA)X = k(AX) = k(\lambda X) = (k\lambda)X$

$\Rightarrow (kA)X = (k\lambda)X$

$\Rightarrow k\lambda$ is an eigen value of kA.

Theorem-3: A and A^T have the same eigen values.

Proof: Let A be any square matrix and A^T denote its transpose. The eigen values of A are the roots of its characteristic polynomial $\det(A - \lambda I) = 0$.

We have $(A - \lambda I)^T = A^T - \lambda I$

Since $\det(A - \lambda I) = \det(A - \lambda I)^T = \det(A^T - \lambda I)$, the characteristic polynomials of A and A^T are the same. Hence the eigen values of A and A^T are same.

Theorem-4: If λ is an eigen value of a square matrix A, then λ^n is an eigen value of A^n .

Proof: Let λ be an eigen value of a square matrix A. Suppose that X is an eigen vector corresponding to λ of A. Then $AX = \lambda X$

$\Rightarrow AAX = A\lambda X$

$\Rightarrow A^2X = \lambda AX = \lambda(\lambda X) = \lambda^2 X$

$\Rightarrow A^2X = \lambda^2 X$

$\Rightarrow \lambda^2$ is an eigen value of A^2

Suppose that λ^{n-1} is an eigen value A^{n-1}

$\Rightarrow A^{n-1}X = \lambda^{n-1} X$

$\Rightarrow A^n X\, A\left(A^{n-1}\right) = A\,\lambda^{n-1} X$

$\Rightarrow A^n X = A\left(A^{n-1} X\right) = A\left(\lambda^{n-1} X\right) = \lambda^{n-1}(AX) = \lambda^{n-1}(\lambda X) = \lambda^n X$

$\Rightarrow A^n X = \lambda^n X$

$\Rightarrow \lambda^n$ is an eigen value of A^n.

Theorem-5: The product of eigen values of square matrix A is the determinant of A and the sum of eigen values is the trace of A.

Proof: Let A be any square matrix of order $n \times n$.

Case 1:

Suppose that $n = 2$ and $A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}_{2\times 2}$

$\Rightarrow \quad A - \lambda I = \begin{bmatrix} a_{11} - \lambda & a_{12} \\ a_{21} & a_{22} - \lambda \end{bmatrix}$

$\Rightarrow \quad \det(A - \lambda I) = (a_{11} - \lambda)(a_{22} - \lambda) - a_{12}\, a_{21}$

$$= a_{11}\, a_{22} - (a_{11} + a_{22})\lambda + \lambda^2 - a_{12}\, a_{21}$$

$$= \lambda^2 - (a_{11} + a_{22})\lambda + (a_{11}\, a_{22} - a_{12}\, a_{21})$$

$$= \lambda^2 - trace\ A\ \lambda + \det A$$

The characteristic equation of A is $\det(A - \lambda I) = 0$.

$\Rightarrow \quad \lambda^2 - trace\ A\ \lambda + \det A = 0 \qquad \rightarrow \quad (1)$

Comparing this equation with $\lambda^2 - (\lambda_1 + \lambda_2)\lambda + \lambda_1\, \lambda_2 = 0$, where λ_1 *and* λ_2 are roots of the characteristic equation of A we get $\lambda_1 + \lambda_2 = trace\ A$ & $\lambda_1\, \lambda_2 = \det A$

$\Rightarrow$ The sum of eigen values of A is the trace of A and the product of eigen values of A is the determinant of A.

Case 2:

Let A be an $n \times n$ square matrix. Suppose that $\lambda_1, \lambda_2, \lambda_3, \ldots, \lambda_n$ are eigen values of A. The characteristic equation of A is $\det(A - \lambda I) = 0$

$$\text{Let} \quad A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \ldots & a_{1n} \\ a_{21} & a_{22} & a_{23} \ldots & a_{2n} \\ \ldots\ldots & \ldots\ldots & \ldots\ldots & \\ a_{n1} & a_{n2} & a_{n3} \ldots & a_{nn} \end{bmatrix}$$

$$A - \lambda I = \begin{bmatrix} a_{11} - \lambda & a_{12} & a_{13} \ldots & a_{1n} \\ a_{21} & a_{22} - \lambda & a_{23} \ldots & a_{2n} \\ \ldots\ldots & \ldots\ldots & \ldots\ldots & \\ a_{n1} & a_{n2} & a_{n3} \ldots & a_{nn} - \lambda \end{bmatrix}$$

$$\det(A - \lambda I) = (-1)^n \lambda^n + (-1)^{n+1}\, trace\ A\ \lambda^{n-1} + \ldots + \det A$$

Now $\det(A - \lambda I) = 0$

$\Rightarrow \quad (-1)^n \lambda^n + (-1)^{n+1}\, trace\ A\ \lambda^{n-1} + \ldots + \det A = 0 \qquad \rightarrow \quad (2)$

Since $\lambda_1, \lambda_2, \lambda_3, \ldots, \lambda_n$ are eigen values of A, they are roots of the characteristic equation $\det(A - \lambda I) = 0$. So we can write

$$(-1)^n \lambda^n + (-1)^{n+1} (\lambda_1 + \lambda_2 + \lambda_3 + \ldots \lambda_n) \lambda^{n-1} + \ldots + (\lambda_1 \lambda_2 \lambda_3 \ldots \lambda_n) = 0 \rightarrow (3)$$

Comparing equations (1) & (2), we get

$$\lambda_1 + \lambda_2 + \lambda_3 + \ldots \lambda_n = trace\ A$$

$$\lambda_1 \lambda_2 \lambda_3 \ldots \lambda_n = \det A$$

$\Rightarrow$ The sum of eigen values of A is the trace of A and the product of eigen values of A is the determinant of A.

Theorem-6: If λ is an eigen value of a non-singular matrix A then $\frac{1}{\lambda}$ is an eigen value of A^{-1}.

Proof: Suppose that A is a non-singular matrix

$\Rightarrow \det A \neq 0$ and A^{-1} exists

Since the product of eigen values of A is the determinant of A, the eigen values of a non-singular matrix cannot be zero. Let λ be an eigen value of A

Since A is non-singular, $\lambda \neq 0$. Let X be an eigen vector of A corresponding to λ.

$\Rightarrow AX = \lambda X$

$\Rightarrow A^{-1}(AX) = A^{-1}(\lambda X)$

$\Rightarrow X = \lambda \left(A^{-1} X\right)$

$\Rightarrow A^{-1} X = \frac{1}{\lambda} X$

$\Rightarrow \frac{1}{\lambda}$ is an eigen value of A^{-1}.

Theorem-7: If λ is an eigen value of a non-singular matrix A, then $\dfrac{\det A}{\lambda}$ is an eigen value of $adj\ A$.

Proof: Suppose that A is a non-singular matrix

$\Rightarrow \det A \neq 0$ and $A^{-1} = \dfrac{1}{\det A}\, adj\ A$ exists

$$\Rightarrow \quad adj\ A = (\det A)\, A^{-1} \qquad \rightarrow \quad (1)$$

Since the product of eigen values of A is the determinant of A, the eigen values of a non-singular matrix cannot be zero. Let λ be an eigen value of A

Since A is non-singular, $\lambda \neq 0$. Let X be an eigen vector of A corresponding to λ.

$\Rightarrow AX = \lambda X$

$\Rightarrow A^{-1}(AX) = A^{-1}(\lambda X)$

$\Rightarrow X = \lambda\left(A^{-1} X\right)$

$$\Rightarrow A^{-1} X = \frac{1}{\lambda} X \qquad \rightarrow (2)$$

Now, $(adj\ \ A)\ X = (\det A)\left(A^{-1} X\right)$

$$= (\det A)\left(\frac{1}{\lambda} X\right)$$

$$= \left(\frac{\det A}{\lambda}\right) X$$

$$\Rightarrow \quad (adj\ \ A)\ X = \left(\frac{\det A}{\lambda}\right) X$$

Hence $\dfrac{\det A}{\lambda}$ is an eigen value of $adj\ A$.

Theorem-8: The eigen values of a diagonal matrix are its principal diagonal entries.

Proof: Let A be a diagonal matrix of order $n \times n$ s follows

$$A = \begin{bmatrix} a_{11} & 0 & 0 \ldots & 0 \\ 0 & a_{22} & 0 \ldots & 0 \\ \ldots\ldots & \ldots\ldots & \ldots\ldots & \ldots \\ 0 & 0 & 0 & \ldots a_{nn} \end{bmatrix}$$

$$\Rightarrow A - \lambda I = \begin{bmatrix} a_{11} - \lambda & 0 & 0 \ldots & 0 \\ 0 & a_{22} - \lambda & 0 \ldots & 0 \\ \ldots\ldots & \ldots\ldots & \ldots\ldots & \ldots \\ 0 & 0 & 0 \ \ldots & a_{nn} - \lambda \end{bmatrix}$$

$\Rightarrow$ det $(A - \lambda I) = (a_{11} - \lambda)(a_{22} - \lambda)(a_{33} - \lambda)\ldots(a_{nn} - \lambda)$

The characteristic equation of A is $\det(A - \lambda I) = 0$

$\Rightarrow (a_{11} - \lambda)(a_{22} - \lambda)(a_{33} - \lambda)\ldots(a_{nn} - \lambda) = 0$

$\Rightarrow$ the roots are a_{11}, a_{22}, a_{33},... and a_{nn}

$\Rightarrow$ the eigen values of A are a_{11}, a_{22}, a_{33} ... and a_{nn}.

Theore-9: The eigen values of a triangular matrix are its principal diagonal entries.

Proof: Let A be a triangular matrix of order $n \times n$ s follows

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \ldots & a_{1n} \\ 0 & a_{22} & a_{23} \ldots & a_{2n} \\ 0 & 0 & a_{33} \ldots & a_{3n} \\ \ldots\ldots & \ldots\ldots & \ldots\ldots & \ldots \\ 0 & 0 & 0 \ \ldots & a_{nn} \end{bmatrix}$$

$$\Rightarrow A-\lambda I=\begin{bmatrix} a_{11}-\lambda & a_{12} & a_{13} & \dots & a_{1n} \\ 0 & a_{22}-\lambda & a_{23} & \dots & a_{2n} \\ 0 & 0 & a_{33}-\lambda & \dots & a_{3n} \\ \dots\dots & \dots\dots & \dots\dots & & \\ 0 & 0 & 0 \dots & a_{nn}-\lambda & \end{bmatrix}$$

$\Rightarrow$ $\det(A-\lambda I)=(a_{11}-\lambda)(a_{22}-\lambda)(a_{33}-\lambda)\ldots(a_{nn}-\lambda)$

The characteristic equation of A is $\det(A-\lambda I)=0$

$\Rightarrow$ $(a_{11}-\lambda)(a_{22}-\lambda)(a_{33}-\lambda)\ldots(a_{nn}-\lambda)=0$

$\Rightarrow$ the roots are a_{11}, a_{22}, a_{33},... and a_{nn}

$\Rightarrow$ the eigen values of A are a_{11}, a_{22}, a_{33} ... and a_{nn}.

Theorem-10: Similar matrices have the same eigen values.

Proof: Suppose that A and B are any two similar matrices. Then there exists a non-singular matrix P such that $B=P^{-1}AP \quad \rightarrow (1)$

We have $\det(B-\lambda I)=\det\left(P^{-1}AP-\lambda P^{-1}IP\right)$

$$= \det\left[P^{-1}(A-\lambda I)P\right]$$

$$= \det P^{-1} \det(A-\lambda I)\det P$$

$$= \det P^{-1}P \det(A-\lambda I)$$

$$= \det I \det(A-\lambda I)$$

$$=\det(A-\lambda I)$$

$\Rightarrow$ $\det(B-\lambda I)=\det(A-\lambda I)$

Thus the characteristic equations B and A are the same.

$\Rightarrow$ the eigen values of B and A are the same.

Theorem-11: If A and B are any two square matrices and if A is invertible then $A^{-1}B$ and BA^{-1} have the same eigen values.

Proof: Suppose that A and B are any two square matrices and A is invertible. Since BA^{-1} and $P^{-1}\left(BA^{-1}\right)P$ are similar matrices, they have same eigen values. Here P is any non-singular matrix. If we take $P = A$ then

$P^{-1}\left(BA^{-1}\right)P = A^{-1}\left(BA^{-1}\right)A = A^{-1}\,B$

$\Rightarrow$ $A^{-1}B$ and BA^{-1} have the same eigen values.

Theorem-12: The eigen values of a Hermitian matrix are real numbers.

Proof: Let A be a Hermitian matrix. That is, $\overline{A^T} = A$

Suppose that X is an eigen vector of A corresponding to an eigen value λ.

$\Rightarrow \quad AX = \lambda X$

$\Rightarrow \quad (AX)^T = (\lambda X)^T$

$\Rightarrow \quad X^T\, A^T = \lambda\, X^T$

$\Rightarrow \quad \overline{X^T}\; \overline{A^T} = \overline{\lambda}\; \overline{X^T}$

$\Rightarrow \quad \overline{X^T}\; A = \overline{\lambda}\; \overline{X^T}$

$\Rightarrow \quad \overline{X^T}\; A\, X = \overline{\lambda}\; \overline{X^T}\; X$

$\Rightarrow \quad \overline{X^T}\; \lambda X = \overline{\lambda}\; \overline{X^T}\; X$

$\Rightarrow \quad \lambda\, \overline{X^T} X = \overline{\lambda}\; \overline{X^T}\; X$

$\Rightarrow \quad \lambda\, \overline{X^T} X - \overline{\lambda}\; \overline{X^T}\; X = 0$

$\Rightarrow \quad \left(\lambda - \overline{\lambda}\right) \overline{X^T}\; X = 0$

Since $\overline{X^T}\; X \neq 0$, we have $\lambda - \overline{\lambda} = 0$

$\Rightarrow \lambda = \overline{\lambda}$

$\Rightarrow \lambda$ is a real number.

Hence The eigen values of a Hermitian matrix are real numbers.

Theorem-13: The eigen values of a skew-Hermitian matrix are purely imaginary or zero.

Proof: Let A be a skew- Hermitian matrix. That is, $\overline{A^T} = -A$

Suppose that X is an eigen vector of A corresponding to an eigen value λ.

$$\Rightarrow \quad AX = \lambda X$$

$$\Rightarrow \quad (AX)^T = (\lambda X)^T$$

$$\Rightarrow \quad X^T \, A^T = \lambda \, X^T$$

$$\Rightarrow \quad \overline{X^T} \, \overline{A^T} = \bar{\lambda} \, \overline{X^T}$$

$$\Rightarrow \quad -\overline{X^T} \, A = \bar{\lambda} \, \overline{X^T}$$

$$\Rightarrow -\overline{X^T} \, A \, X = \bar{\lambda} \, \overline{X^T} \, X$$

$$\Rightarrow \quad -\overline{X^T} \, \lambda X = \bar{\lambda} \, \overline{X^T} \, X$$

$$\Rightarrow \quad -\lambda \, \overline{X^T} X = \bar{\lambda} \, \overline{X^T} \, X$$

$$\Rightarrow \quad \lambda \, \overline{X^T} X + \bar{\lambda} \, \overline{X^T} \, X = 0$$

$$\Rightarrow \quad \left(\lambda + \bar{\lambda}\right) \overline{X^T} \, X = 0$$

Since $\overline{X^T} \, X \neq 0$, we have $\lambda + \bar{\lambda} = 0$

$\operatorname{Re} \lambda = 0$ where $\operatorname{Re} \lambda$ denotes the real part of λ

$\Rightarrow \lambda = 0$ or λ is purely imaginary.

Hence The eigen values of a skew-Hermitian matrix are purely imaginary or zero.

Theorem-14: The eigen values of a Unitary matrix have absolute value one.

Proof: Let A be a Unitary matrix. That is, $A\,\overline{A^T} = \overline{A^T}\,A = I$

Suppose that X is an eigen vector of A corresponding to an eigen value λ.

$\Rightarrow \quad AX = \lambda X$

$\Rightarrow \quad (AX)^T = (\lambda X)^T$

$\Rightarrow \quad X^T\,A^T = \lambda\,X^T$

$\Rightarrow \quad \overline{X^T}\,\overline{A^T} = \overline{\lambda}\,\overline{X^T}$

$\Rightarrow \quad \overline{X^T}\,\overline{A^T}\,A = \overline{\lambda}\,\overline{X^T}\,A$

$\Rightarrow \quad \overline{X^T}\,I = \overline{\lambda}\,\overline{X^T}\,A$

$\Rightarrow \quad \overline{X^T} = \overline{\lambda}\,\overline{X^T}\,A$

$\Rightarrow \quad \overline{X^T}\,X = \overline{\lambda}\,\overline{X^T}\,AX$

$\Rightarrow \quad \overline{X^T}\,X = \overline{\lambda}\,\overline{X^T}\,\lambda X$

$\Rightarrow \quad \overline{X^T}\,X = \lambda\,\overline{\lambda}\,\overline{X^T}\,X$

$\Rightarrow \quad \lambda\,\overline{\lambda}\,\overline{X^T}\,X - \overline{X^T}\,X = 0$

$\Rightarrow \quad \left(\lambda\,\overline{\lambda} - 1\right)\overline{X^T}\,X = 0$

Since $\overline{X^T}\,X \neq 0$, we have $\lambda\,\overline{\lambda} - 1 = 0$

$\Rightarrow \lambda\,\overline{\lambda} = 1$

$\Rightarrow \quad |\lambda|^2 = 1$

$\Rightarrow \quad |\lambda| = 1$

Hence The eigen values of a Unitary matrix have absolute value one.

Properties of Eigen Vectors

Theorem-1: If X_1 and X_2 are any two eigen vectors of a square matrix A corresponding to an eigen value λ then $X_1 + X_2$ is also an eigen vector of A corresponding to λ.

Proof: Suppose that X_1 and X_2 are any two eigen vectors of a square matrix A corresponding to an eigen value λ. Then $AX_1 = \lambda X_1 \quad \to$ (1)

$$AX_2 = \lambda X_2 \quad \to (2)$$

$$\Rightarrow A(X_1 + X_2) = AX_1 + AX_2$$

$$= \lambda X_1 + \lambda X_2$$

$$= \lambda(X_1 + X_2)$$

$$\Rightarrow A(X_1 + X_2) = \lambda(X_1 + X_2)$$

$\Rightarrow X_1 + X_2$ is an eigen vector of A corresponding to λ.

Theorem-2: If k is a constant and X is an eigen vector of a square matrix A corresponding to eigen value λ then kX is also eigen vector of A corresponding to λ.

Proof: Suppose that k is a constant and X is an eigen vector of a square matrix A corresponding to eigen value λ. Then $AX = \lambda X \Rightarrow kAX = k\lambda X$

$$\Rightarrow A(kX) = \lambda(kX)$$

$\Rightarrow$ kX is eigen vector of A corresponding to λ.

Theorem-3: Two eigen vectors of a square matrix A corresponding to two different eigen values are linearly independent.

Proof: Suppose that X_1 and X_2 are any two eigen vectors of a square matrix A corresponding to two different eigen values λ_1 and λ_2 respectively.

$$\Rightarrow AX_1 = \lambda_1 X_1 \quad \text{and} \quad AX_2 = \lambda_2 X_2$$

Suppose that $c_1X_1 + c_2X_2 = O \quad \to$ (1) where c_1 and c_2 are constants

$\Rightarrow A\left(c_1X_1 + c_2X_2\right) = O$

$\Rightarrow c_1 \left(AX_1\right) + c_2 \left(AX_2\right) = O$

$\Rightarrow c_1 \left(\lambda_1X_1 \right) + c_2 \left(\lambda_2 X_2 \right) = O \qquad \rightarrow (2)$

Solving (1) and (2)

$c_1\left(\lambda_1 - \lambda_2\right) X_1 = O$

$\Rightarrow c_1 = 0$

Similarly $c_2 \left(\lambda_2 - \lambda_1\right) X_2 = O$

$\Rightarrow c_2 = 0$

Hence X_1 and X_2 are linearly independent.

Definition: Two eigen vectors X_1 and X_2 are said to be orthogonal if $X_1^T X_2 = 0$ or $X_2^T X_1 = 0$.

Theorem-4: Two eigen vectors of a symmetric matrix A corresponding to two different eigen values are orthogonal.

Proof: Suppose that X_1 and X_2 are any two eigen vectors of a symmetric matrix A corresponding to two different eigen values λ_1 and λ_2 respectively.

$\Rightarrow AX_1 = \lambda_1X_1$ and $AX_2 = \lambda_2X_2$

$\Rightarrow X_2^TAX_1 = X_2^T\lambda_1X_1$ and $AX_2 = \lambda_2X_2$

$\Rightarrow \left(X_2^TAX_1\right)^T = \left(X_2^T\lambda_1X_1\right)^T$ and $X_1^TAX_2 = X_1^T\lambda_2X_2$

$\Rightarrow X_1^TA^TX_2 = \lambda_1 X_1^T X_2$ and $X_1^TAX_2 = \lambda_2X_1^TX_2$

Since A is symmetric, $A^T = A$.

Consider $X_1^TA^TX_2 = \lambda_1 X_1^T X_2$ and $X_1^TAX_2 = \lambda_2X_1^TX_2$

$\Rightarrow X_1^TAX_2 = \lambda_1 X_1^T X_2$ and $X_1^TAX_2 = \lambda_2X_1^TX_2$

$\Rightarrow \lambda_1 X_1^T X_2 = \lambda_2 X_1^T X_2$

$\Rightarrow \lambda_1 X_1^T X_2 - \lambda_2 X_1^T X_2 = 0$

$\Rightarrow (\lambda_1 - \lambda_2) X_1^T X_2 = 0$

$\Rightarrow \lambda_1 - \lambda_2 = 0 \quad or \quad X_1^T X_2 = 0$

Since $\lambda_1 \neq \lambda_2$, $\lambda_1 - \lambda_2 \neq 0$

$\Rightarrow X_1^T X_2 = 0$

$\Rightarrow X_1$ and X_2 are orthogonal.

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Cayley – Hamilton theorem

Statement: Every square matrix satisfies its own characteristic equation.

Problem: Verify Cayley- Hamilton theorem for the matrix $A = \begin{bmatrix} 2 & 1 & 2 \\ 5 & 3 & 3 \\ -1 & 0 & -2 \end{bmatrix}$.

Solution: Given $A = \begin{bmatrix} 2 & 1 & 2 \\ 5 & 3 & 3 \\ -1 & 0 & -2 \end{bmatrix}$

$$\Rightarrow \quad A - \lambda I = \begin{bmatrix} 2-\lambda & 1 & 2 \\ 5 & 3-\lambda & 3 \\ -1 & 0 & -2-\lambda \end{bmatrix}$$

$$\Rightarrow \quad \det(A-\lambda I) = (2-\lambda)\left[(3-\lambda)(-2-\lambda)\right] - \left[5(-2-\lambda)+3\right] + 2[\ 3-\lambda]$$

$$\Rightarrow \quad \lambda^3 - 3\lambda^2 - 7\lambda - 1$$

Then the characteristic equation of A is $\det(A-\lambda I) = 0$

$$\Rightarrow \quad \lambda^3 - 3\lambda^2 - 7\lambda - 1 = 0 \qquad \rightarrow \quad (1)$$

Now $A^2 = \begin{bmatrix} 2 & 1 & 2 \\ 5 & 3 & 3 \\ -1 & 0 & -2 \end{bmatrix} \begin{bmatrix} 2 & 1 & 2 \\ 5 & 3 & 3 \\ -1 & 0 & -2 \end{bmatrix} = \begin{bmatrix} 7 & 5 & 3 \\ 22 & 14 & 13 \\ 0 & -1 & 2 \end{bmatrix}$

$$\Rightarrow A^3 = \begin{bmatrix} 7 & 5 & 3 \\ 22 & 14 & 13 \\ 0 & -1 & 2 \end{bmatrix} \begin{bmatrix} 2 & 1 & 2 \\ 5 & 3 & 3 \\ -1 & 0 & -2 \end{bmatrix} = \begin{bmatrix} 36 & 22 & 23 \\ 101 & 64 & 60 \\ -7 & -3 & -7 \end{bmatrix}$$

Then

$$\text{LHS} = \begin{bmatrix} 36 & 22 & 23 \\ 101 & 64 & 60 \\ -7 & -3 & -7 \end{bmatrix} - \begin{bmatrix} 21 & 15 & 9 \\ 66 & 42 & 39 \\ 0 & -3 & 6 \end{bmatrix} - \begin{bmatrix} 14 & 7 & 14 \\ 35 & 21 & 21 \\ -7 & 0 & -14 \end{bmatrix} - \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 15 & 7 & 14 \\ 35 & 22 & 21 \\ -7 & 0 & -13 \end{bmatrix} - \begin{bmatrix} 14 & 7 & 14 \\ 35 & 21 & 21 \\ -7 & 0 & -14 \end{bmatrix} - \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} - \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} = O$$

= RHS

$\Rightarrow \ \lambda^3 - 3\lambda^2 - 7\lambda - 1 = O$.

$\Rightarrow$ A satisfies its characteristic equation.

Problem: Find A^{-1} of a matrix $A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & -1 & 4 \\ 3 & 1 & -1 \end{bmatrix}$ by using Cayley – Hamilton theorem.

Solution: Given $A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & -1 & 4 \\ 3 & 1 & -1 \end{bmatrix}$

$$\Rightarrow \ A - \lambda I = \begin{bmatrix} 1-\lambda & 2 & 3 \\ 2 & -1-\lambda & 4 \\ 3 & 1 & -1-\lambda \end{bmatrix}$$

$$\Rightarrow \ \det(A - \lambda I) = (1-\lambda)\left[(-1-\lambda)^2 - 4\right] - 2\left[2(-1-\lambda) - 12\right] - 3\left[2 - 3(-1-\lambda)\right]$$

$$= \lambda^3 + \lambda^2 - 18\lambda - 40$$

The characteristic equation of A is $\det(A - \lambda I) = 0$

$\Rightarrow \ \lambda^3 + \lambda^2 - 18\lambda - 40 = 0 \qquad \rightarrow (1)$

By the Cayley – Hamilton theorem, A satisfies its characteristic equation

$\Rightarrow \ A^3 + A^2 - 18A - 40\,I = O \qquad \rightarrow (2)$

Multiplying with A^{-1}, we get $\quad A^2 + A - 18I - 40\,A^{-1} = O$

$$\Rightarrow \quad 40\,A^{-1} = A^2 + A - 18I$$

Now $$A^2 = \begin{bmatrix} 1 & 2 & 3 \\ 2 & -1 & 4 \\ 3 & 1 & -1 \end{bmatrix} \begin{bmatrix} 1 & 2 & 3 \\ 2 & -1 & 4 \\ 3 & 1 & -1 \end{bmatrix} = \begin{bmatrix} 14 & 3 & 8 \\ 12 & 9 & -2 \\ 2 & 4 & 14 \end{bmatrix}$$

From equation (2), we have

$$40\,A^{-1} = \begin{bmatrix} 14 & 3 & 8 \\ 12 & 9 & -2 \\ 2 & 4 & 14 \end{bmatrix} + \begin{bmatrix} 1 & 2 & 3 \\ 2 & -1 & 4 \\ 3 & 1 & -1 \end{bmatrix} - 18 \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} -3 & 5 & 11 \\ 14 & -10 & 2 \\ 5 & 5 & -5 \end{bmatrix}$$

$$\Rightarrow \quad A^{-1} = \frac{1}{40} \begin{bmatrix} -3 & 5 & 11 \\ 14 & -10 & 2 \\ 5 & 5 & -5 \end{bmatrix}$$

Problem: Find A^4 of a matrix $A = \begin{bmatrix} 7 & 2 & -2 \\ -6 & -1 & 2 \\ 6 & 2 & -1 \end{bmatrix}$ by using Cayley – Hamilton theorem.

Solution: Given $A = \begin{bmatrix} 7 & 2 & -2 \\ -6 & -1 & 2 \\ 6 & 2 & -1 \end{bmatrix}$

$$\Rightarrow \quad A - \lambda I = \begin{bmatrix} 7-\lambda & 2 & -2 \\ -6 & -1-\lambda & 2 \\ 6 & 2 & -1-\lambda \end{bmatrix}$$

$$\Rightarrow \quad \det(A - \lambda I) = (7-\lambda)\left[(-1-\lambda)^2 - 4\right] - 2\left[-6(-1-\lambda) - 12\right] - 2\left[-12 - 6(-1-\lambda)\right]$$

$$= \lambda^3 - 5\,\lambda^2 + 7\lambda - 3$$

The characteristic equation of A is $\det(A - \lambda I) = 0$

$$\Rightarrow \quad \lambda^3 - 5\lambda^2 + 7\lambda - 3 = 0$$

By the Cayley – Hamilton theorem, A satisfies its characteristic equation

$$\Rightarrow A^3 - 5A^2 + 7A - 3I = O$$

Multiplying with A, we get $\quad A^4 - 5A^3 + 7A^2 - 3A = O$

$$\Rightarrow \quad A^4 = 5A^3 - 7A^2 + 3A$$

Now $A^2 = \begin{bmatrix} 7 & 2 & -2 \\ -6 & -1 & 2 \\ 6 & 2 & -1 \end{bmatrix} \begin{bmatrix} 7 & 2 & -2 \\ -6 & -1 & 2 \\ 6 & 2 & -1 \end{bmatrix} = \begin{bmatrix} 25 & 8 & -8 \\ -24 & -7 & 8 \\ 24 & 8 & -7 \end{bmatrix}$

$$A^3 = \begin{bmatrix} 25 & 8 & -8 \\ -24 & -7 & 8 \\ 24 & 8 & -7 \end{bmatrix} \begin{bmatrix} 7 & 2 & -2 \\ -6 & -1 & 2 \\ 6 & 2 & -1 \end{bmatrix} = \begin{bmatrix} 79 & 26 & -26 \\ -78 & -25 & 26 \\ 78 & 26 & -25 \end{bmatrix}$$

Then

$$\text{LHS} = 5\begin{bmatrix} 79 & 26 & -26 \\ -78 & -25 & 26 \\ 78 & 26 & -25 \end{bmatrix} - 7\begin{bmatrix} 25 & 8 & -8 \\ -24 & -7 & 8 \\ 24 & 8 & -7 \end{bmatrix} + 3\begin{bmatrix} 7 & 2 & -2 \\ -6 & -1 & 2 \\ 6 & 2 & -1 \end{bmatrix}$$

$$= \begin{bmatrix} 395 & 130 & -130 \\ -240 & -125 & 130 \\ 390 & 130 & -125 \end{bmatrix} - \begin{bmatrix} 175 & 56 & -56 \\ -168 & -49 & 56 \\ 168 & 56 & -49 \end{bmatrix} + \begin{bmatrix} 21 & 6 & -6 \\ -18 & -3 & 6 \\ 18 & 6 & -3 \end{bmatrix}$$

$$= \begin{bmatrix} 241 & 80 & -80 \\ -240 & -79 & 80 \\ 240 & 80 & -79 \end{bmatrix}$$

Exercise

1. Verify the Cayley-Hamilton theorem for the following matrices.

a. $\begin{bmatrix} 3 & 1 & 4 \\ 0 & 2 & 6 \\ 0 & 0 & 5 \end{bmatrix}$ b. $\begin{bmatrix} 1 & 0 & -1 \\ 1 & 2 & 1 \\ 2 & 2 & 3 \end{bmatrix}$ c. $\begin{bmatrix} 1 & -2 & 2 \\ 1 & -2 & 3 \\ 0 & -1 & 2 \end{bmatrix}$ d. $\begin{bmatrix} 1 & 2 & 3 \\ 2 & 4 & 5 \\ 3 & 5 & 6 \end{bmatrix}$

2. Using Cayley – Hamilton theorem, find the inverse of the following matrices.

a. $\begin{bmatrix} 7 & -1 & 3 \\ 6 & 1 & 4 \\ 2 & 4 & 8 \end{bmatrix}$ b. $\begin{bmatrix} 4 & 6 & 6 \\ 1 & 3 & 2 \\ -1 & -4 & -3 \end{bmatrix}$ c. $\begin{bmatrix} 2 & -1 & 1 \\ -1 & 2 & -1 \\ 1 & -1 & 2 \end{bmatrix}$ d. $\begin{bmatrix} 1 & 1 & 2 \\ 3 & 1 & 1 \\ 2 & 3 & 1 \end{bmatrix}$

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Diagonalization

Definition: A square matrix A is said to be *diagonalizable*, if there exists a non – singular matrix P such that $P^{-1}A\,P$ is a diagonal matrix. That is, A is said to be diagonalizable if it is singular to a diagonal matrix. The non – singular matrix P which diagonalizes A is called a *modal matrix* and the diagonal matrix $D = P^{-1}A\,P$ is called a *spectral matrix*.

Remarks: Let A be a square matrix of order n.

1. If A has n distinct eigen values then A is diagonalizable.
2. If A has exactly n linearly independent eigen vectors then A is diagonalizable.
3. Every symmetric matrix is diagonalizable.

Problem: Diagonalize the matrix $A = \begin{bmatrix} -5 & 2 \\ 2 & -2 \end{bmatrix}$

Solution: Given $A = \begin{bmatrix} -5 & 2 \\ 2 & -2 \end{bmatrix}$. Then $A - \lambda I = \begin{bmatrix} -5-\lambda & 2 \\ 2 & -2-\lambda \end{bmatrix}$

The characteristic polynomial of A is $\det(A - \lambda I) = 0$

$\Rightarrow (-5-\lambda)(-2-\lambda) - 4 = 0$

$\Rightarrow (5+\lambda)(2+\lambda) + 4 = 0$

$\Rightarrow \lambda^2 + 7\lambda + 6 = 0$

$\Rightarrow (\lambda+1)(\lambda+6) = 0$

$\Rightarrow \lambda = -1 \text{ or } -6$. Put $\lambda_1 = -1$ and $\lambda_2 = -6$

Then $\lambda_1 = -1$ and $\lambda_2 = -6$ are eigen values of the given matrix A.

First we find eigen vectors corresponding to $\lambda_1 = -1$. For this, consider $(A - \lambda I)X = O$. Put $\lambda = \lambda_1 = -1$

$\Rightarrow (A + I)X = O$

$\Rightarrow \begin{bmatrix} -4 & 2 \\ 2 & -1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$, where $x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$

Now $(A + I) = \begin{bmatrix} -4 & 2 \\ 2 & -1 \end{bmatrix}$

Apply $R_2 \to 2R_2 + R_1$

$$(A+I) \sim \begin{bmatrix} -4 & 2 \\ 0 & 0 \end{bmatrix}$$

Take $x_2 = k$, where k is a constant

From the first row of $(A+I) \sim \begin{bmatrix} -4 & 2 \\ 0 & 0 \end{bmatrix}$, we have $-4x_1 + 2x_2 = 0$

$-2x_1 + x_2 = 0 \Rightarrow x_1 = \dfrac{k}{2}$

$X = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} \dfrac{k}{2} \\ k \end{bmatrix}$ is a general solution of $(A+I)X = O$

$\Rightarrow X_1 = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$, $X_2 = \begin{bmatrix} -1 \\ -2 \end{bmatrix}$, $X_3 = \begin{bmatrix} 2 \\ 4 \end{bmatrix}, \ldots$ are eigen vectors corresponding to $\lambda_1 = -1$

Now we find eigen vectors corresponding to $\lambda_2 = -6$

Consider $(A - \lambda I)X = O$ and put $\lambda = \lambda_2 = -6$

$\Rightarrow (A + 6I)X = O$

$\Rightarrow \begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$, where $x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$

Take $(A+I) = \begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix}$

Apply $R_2 \to R_2 - 2R_1$

$$(A + 6\,I) \sim \begin{bmatrix} 1 & 2 \\ 0 & 0 \end{bmatrix}$$

Choose $x_2 = k$, where k is a constant

From the first row of $(A + 6\,I) \sim \begin{bmatrix} 1 & 2 \\ 0 & 0 \end{bmatrix}$, we have $x_1 + 2x_2 = 0$

$\Rightarrow x_1 = -2k$

$\Rightarrow Y = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} -2k \\ k \end{bmatrix}$ is a general solution of $(A + 6I)X = O$

$\Rightarrow \quad Y_1 = \begin{bmatrix} -2 \\ 1 \end{bmatrix}, \quad Y_2 = \begin{bmatrix} 2 \\ -1 \end{bmatrix}, \quad Y_3 = \begin{bmatrix} -4 \\ 2 \end{bmatrix}, \ldots$ are eigen vectors corresponding to $\lambda_2 = -6$.

Consider $X_1 = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$ and $Y_1 = \begin{bmatrix} -2 \\ 1 \end{bmatrix}$. These two vectors are linearly independent.

Let $P = \begin{bmatrix} 1 & -2 \\ 2 & 1 \end{bmatrix}$. This matrix P is non – singular and $P^{-1} = \frac{1}{5}\begin{bmatrix} 1 & 2 \\ -2 & 1 \end{bmatrix}$.

Then $D = P^{-1} A P = \begin{bmatrix} -1 & 0 \\ 0 & -6 \end{bmatrix}$.

This diagonal matrix D is similar to A and A diagonalizable.

Problem: Diagonalize $A = \begin{bmatrix} -9 & 4 & 4 \\ -8 & 3 & 4 \\ -16 & 8 & 7 \end{bmatrix}$.

Solution: Given matrix is $A = \begin{bmatrix} -9 & 4 & 4 \\ -8 & 3 & 4 \\ -16 & 8 & 7 \end{bmatrix}$

Then $A - \lambda I = \begin{bmatrix} -9-\lambda & 4 & 4 \\ -8 & 3-\lambda & 1 \\ -16 & 8 & 7-\lambda \end{bmatrix}$

The characteristic equation of A is $\det(A - \lambda I) = 0$.

$\Rightarrow \lambda^3 - \lambda^2 - 5\lambda - 3 = 0$

$\Rightarrow (\lambda + 1)^2 (\lambda - 3) = 0$

$\Rightarrow \lambda = -1, -1 \text{ or } 3$

$\Rightarrow \lambda_1 = -1 \text{ and } \lambda_2 = 3$ are eigen values of A.

To find eigen vectors corresponding to $\lambda_1 = -1$, consider $(A - \lambda_1 I)X = O$.

Now $A - \lambda_1 I = A + I = A = \begin{bmatrix} -8 & 4 & 4 \\ -8 & 4 & 4 \\ -16 & 8 & 8 \end{bmatrix}$

Apply $R_2 \to R_2 - R_1$ *and* $R_3 \to R_3 - 2R_1$

$$A + I \sim \begin{bmatrix} -8 & 4 & 4 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

Apply $R_1 \to \frac{R_1}{4}$

$$A + I \sim \begin{bmatrix} -2 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

Take $x_2 = k_1$ and $x_3 = k_2$ where k_1 *and* k_2 are constants

From the first row, we have $-2x_1 + x_2 + x_3 = 0$

$$\Rightarrow x_1 = \frac{k_1 + k_2}{2}$$

Then

$\begin{bmatrix} \frac{k_1 + k_2}{2} \\ k_1 \\ k_2 \end{bmatrix}$ is a general solution of $(A - \lambda_1 I)X = O$

Choose the eigen vectors $X_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$ *and* $X_2 = \begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix}$ corresponding to $\lambda_1 = -1$.

Now we find eigen vectors corresponding to $\lambda_2 = 3$

Consider $(A - \lambda_2 I)X = O$

Then $A - \lambda_2 I = A - 3I = A = \begin{bmatrix} -12 & 4 & 4 \\ -8 & 0 & 4 \\ -16 & 8 & 4 \end{bmatrix}$

Apply $R_1 \to \frac{R_1}{4}$, $R_2 \to \frac{R_2}{4}$ *and* $R_3 \to \frac{R_3}{4}$

$$A - 3I \sim \begin{bmatrix} -3 & 1 & 1 \\ -2 & 0 & 1 \\ -4 & 2 & 1 \end{bmatrix}$$

Apply $R_3 \to R_3 - 2\,R_2$

$$A - 3I \sim \begin{bmatrix} -3 & 1 & 1 \\ -2 & 0 & 1 \\ 0 & 2 & -1 \end{bmatrix}$$

Apply $R_2 \to 3R_2 - 2\,R_1$

$$A - 3I \sim \begin{bmatrix} -3 & 1 & 1 \\ 0 & -2 & 1 \\ 0 & 2 & -1 \end{bmatrix}$$

Apply $R_3 \to R_3 + R_2$

$$A - 3I \sim \begin{bmatrix} -3 & 1 & 1 \\ 0 & -2 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

Take $x_3 = k$ where k is a constant

Then $-2x_2 + x_3 = 0$

$$\Rightarrow \quad x_2 = \frac{k}{2}$$

$$-3x_1 + x_2 + x_3 = 0 \;\Rightarrow x_1 = \frac{k}{2}$$

Then

$$\Rightarrow \begin{bmatrix} \frac{k}{2} \\ \frac{k}{2} \\ k \end{bmatrix} \quad \text{is a general solution of } \left(A - \lambda_2 I\right)X = O$$

Choose the eigen vector $X_3 = \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix}$ corresponding to $\lambda_2 = 3$.

Suppose that $c_1X_1 + c_2X_2 + c_3X_3 = O$

$$\Rightarrow \quad c_1 \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} + c_2 \begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix} + c_3 \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$\Rightarrow \; c_1 + c_3 = 0, \quad c_1 - c_2 + c_3 = 0 \;$ *and* $c_1 + c_2 + 2c_3 = 0$

$\Rightarrow \; c_1 = 0 \, , \; c_2 = 0 \, , c_3 = 0$

$\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} , \begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix}$ *and* $\begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix}$ are linearly independent vectors

The modal matrix is given by $P = \begin{bmatrix} 1 & 0 & 1 \\ 1 & -1 & 1 \\ 1 & 1 & 2 \end{bmatrix}$

$$\Rightarrow \; P^{-1} = \begin{bmatrix} 3 & -1 & -1 \\ 1 & -1 & 0 \\ -2 & 1 & 1 \end{bmatrix}$$

$$\Rightarrow \; P^{-1}AP = \begin{bmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 3 \end{bmatrix}$$

$\Rightarrow \; A$ is diagonalizable and $D = P^{-1}AP$ is a diagonal matrix similar to A

Example: The matrix $A = \begin{bmatrix} 0 & -6 & -4 \\ 5 & -11 & -6 \\ -6 & 9 & 4 \end{bmatrix}$ is not diagonalizable.

Exercise

1. Diagonalize the following matrices.

a. $\begin{bmatrix} 2 & 1 & -1 \\ 1 & 4 & 3 \\ -1 & 3 & 4 \end{bmatrix}$

b. $\begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$

c. $\begin{bmatrix} 2 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 1 & 1 \end{bmatrix}$

d. $\begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 1 \\ 1 & 1 & 1 \end{bmatrix}$

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Quadratic Forms

Definition: A *quadratic form* in n variables $(x_1, x_2, ..., x_n)$ is defined by

$$X^T AX \quad \text{where } A = [a_{ij}]_{n\times n} \quad \text{and} \quad X = \begin{bmatrix} x_1 \\ x_2 \\ . \\ . \\ x_n \end{bmatrix}.$$

The matrix $A = [a_{ij}]_{n\times n}$ is called a matrix of the quadratic form

Problem: Write a quadratic form in two variables x_1 and x_2.

Solution: Let $A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$ and $X = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$

Then $X^T AX = [x_1 \quad x_2] \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$

$$= a_{11}x_1^2 + [a_{12} + a_{21}]\, x_1x_2 + a_{22}x_2^2$$

$\Rightarrow \; X^T AX = a_{11}x_1^2 + [a_{12} + a_{21}]\, x_1x_2 + a_{22}x_2^2$

If we take $a_{12} = a_{21}$, then $X^T AX = a_{11}x_1^2 + 2a_{12}\, x_1x_2 + a_{22}x_2^2$

In this case $a_{12} = a_{21}$, $A = \begin{bmatrix} a_{11} & a_{12} \\ a_{12} & a_{22} \end{bmatrix}$ and then A is symmetric.

Problem: Write a quadratic form in three variables x_1, x_2 and x_3.

Solution: Let $A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$ and $X = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$

Then $X^T AX = [x_1 \quad x_2 \quad x_3] \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$

$= a_{11}x_1^2 + [a_{12} + a_{21}]\, x_1x_2 + a_{22}x_2^2 + [a_{23} + a_{32}]\, x_2x_3 + [a_{13} + a_{31}]\, x_1x_3 + a_{33}x_3^2$

$\Rightarrow\ X^T AX = a_{11}x_1^2 + a_{22}x_2^2 + a_{33}x_3^2 + [a_{12} + a_{21}]\, x_1x_2 + [a_{23} + a_{32}]\, x_2x_3 + [a_{13} + a_{31}]\, x_1x_3$

If we take $a_{12} = a_{21}$, $a_{13} = a_{31}$ and $a_{23} = a_{32}$ then

$X^T AX = a_{11}x_1^2 + a_{22}x_2^2 + a_{33}x_3^2 + 2a_{12}x_1x_2 + 2a_{23}x_2x_3 + 2a_{13}x_1x_3$

In this case, $A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{12} & a_{22} & a_{23} \\ a_{13} & a_{23} & a_{33} \end{bmatrix}$ and then A is symmetric.

Problem: Write a symmetric matrix corresponding to the quadratic form $2x_1^2 - x_2^2 + 6x_1x_2$

Solution: Given quadratic form is $2x_1^2 - x_2^2 + 6x_1x_2$

Comparing with $X^T AX = a_{11}x_1^2 + 2a_{12}\, x_1x_2 + a_{22}x_2^2$, we get

$a_{11} = 2$, $a_{22} = -1$ *and* $a_{12} = a_{21} = 3$

Then the symmetric matrix of the quadratic form is $A = \begin{bmatrix} 2 & 3 \\ 3 & -1 \end{bmatrix}$.

Problem: Write a quadratic form corresponding to the matrix $A = \begin{bmatrix} 2 & 2 & 1 \\ 1 & 3 & 1 \\ 1 & 2 & 2 \end{bmatrix}$.

Solution: Given $A = \begin{bmatrix} 2 & 2 & 1 \\ 1 & 3 & 1 \\ 1 & 2 & 2 \end{bmatrix}$. Let $X = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$

$\Rightarrow\ X^T AX = a_{11}x_1^2 + a_{22}x_2^2 + a_{33}x_3^2 + [a_{12} + a_{21}]\, x_1x_2 + [a_{23} + a_{32}]\, x_2x_3 + [a_{13} + a_{31}]\, x_1x_3$

$\Rightarrow\ X^T AX = 2x_1^2 + 3x_2^2 + 2x_3^2 + 3x_1x_2 + 3x_2x_3 + 2x_1x_3$.

Problem: Write a symmetric matrix corresponding to the quadratic form

$2x_1^2 + 4x_2^2 + 4x_3^2 + 2x_1x_2 + 6x_2x_3 - 2x_1x_3$

Solution: Given quadratic form is $2x_1^2+4x_2^2+4x_3^2+2x_1x_2+6x_2x_3-2x_1x_3$

Comparing with $a_{11}x_1^2+a_{22}x_2^2+a_{33}x_3^2+2a_{12}x_1x_2+2a_{23}x_2x_3+2a_{13}x_1x_3$, we get

$a_{11}=2$, $a_{22}=4$, $a_{33}=4$, $a_{12}=a_{21}=1$, $a_{23}=a_{23}=3$ *and* $a_{13}=a_{31}=-1$

Then the symmetric matrix corresponding to the given quadratic form is

$$\begin{bmatrix} 2 & 1 & -1 \\ 1 & 4 & 3 \\ -1 & 3 & 4 \end{bmatrix}$$

Definition: The *normal form* or canonical form of a quadratic form X^TAX in n variables is defined by $\lambda_1x_1^{\,2}+\lambda_2x_2^{\,2}+\lambda_3x_3^{\,2}+\ldots+\lambda_nx_n^{\,2}$.

Definition: Let X^TAX be a quadratic form in n variables and let A be the symmetric matrix corresponding to X^TAX. Then

1. The *rank* of the quadratic form X^TAX is defined to be the rank of A. It is denoted by $\rho(A)=r$.
2. The number of positive terms in the normal form of X^TAX is called the *index* of the quadratic form. It is denoted by s.
3. The *signature* of X^TAX is defined by $2s-r$.

Definition: A quadratic form X^TAX in n variables is said to be

1. *Positive definite* if $r=s=n$ or if all the eigen values of A are positive
2. *Positive semidefinite* if $r=s<n$ or if the eigen values of A are zero or positive
3. *Negative definite* if $r=n$ and $s=0$ or if all the eigen values of A are negative
4. *Negative semidefinite* if $r<n$ and $s=0$ or if the eigen values of A are zero or negative.

Problem: Reduce the quadratic form

$8x_1^2 + 7x_2^2 + 3x_3^2 - 12x_1x_2 - 8x_2x_3 + 4x_1x_3$ into normal form and hence find its rank, signature and index.

Solution: Given quadratic form is

$$X^T AX = 8x_1^2 + 7x_2^2 + 3x_3^2 - 12x_1x_2 - 8x_2x_3 + 4x_1x_3$$

Comparing with

$X^T AX = a_{11}x_1^2 + a_{22}x_2^2 + a_{33}x_3^2 + 2a_{12}x_1x_2 + 2a_{23}x_2x_3 + 2a_{13}x_1x_3$, we get

$a_{11} = 8,\ a_{22} = 7,\ a_{33} = 3,\ a_{12} = a_{21} = -6,\ a_{13} = a_{31} = 2$ and $a_{23} = a_{32} = -4$

Now the symmetric matrix corresponding to the given quadratic form is

$$A = \begin{bmatrix} 8 & -6 & 2 \\ -6 & 7 & -4 \\ 2 & -4 & 3 \end{bmatrix}$$

$$\Rightarrow A - \lambda I = \begin{bmatrix} 8-\lambda & -6 & 2 \\ -6 & 7-\lambda & -4 \\ 2 & -4 & 3-\lambda \end{bmatrix}$$

The characteristic equation of A is $\lambda^3 - 18\lambda^2 + 45\lambda = 0$

$$\Rightarrow \lambda(\lambda-3)(\lambda-15) = 0$$

Hence $\lambda_1 = 0, \lambda_2 = 3$ and $\lambda_3 = 15$ are the eigen values of A.

To find the eigen vectors corresponding to $\lambda_1 = 0$, consider $(A - \lambda_1 I)X = O$

$$\Rightarrow AX = O$$

Consider $A = \begin{bmatrix} 8 & -6 & 2 \\ -6 & 7 & -4 \\ 2 & -4 & 3 \end{bmatrix}$

Apply $R_2 \to 4R_2 + 3R_1$ and $R_3 \to 4R_3 - R_1$

$$A \sim \begin{bmatrix} 8 & -6 & 2 \\ 0 & 10 & -10 \\ 0 & -10 & 10 \end{bmatrix}$$

Apply $R_3 \to R_3 + R_2$

$$A \sim \begin{bmatrix} 8 & -6 & 2 \\ 0 & 10 & -10 \\ 0 & 0 & 0 \end{bmatrix}$$

Take $x_3 = k$, where k is a constant

Now $10x_2 - 10x_3 = 0 \Rightarrow x_2 = k$

Also $8x_1 - 6x_2 + 2x_3 = 0 \Rightarrow x_1 = \dfrac{k}{2}$

Therefore $\begin{bmatrix} \frac{k}{2} \\ k \\ k \end{bmatrix}$ is a general solution of $(A - \lambda_1 I)X = O$

Choose the eigen vector $X_1 = \begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix}$ corresponding to $\lambda_1 = 0$.

To find eigen vectors corresponding to $\lambda_2 = 3$, consider $(A - \lambda_2 I)X = O$

We have $A - \lambda_2 I = A - 3I = \begin{bmatrix} 5 & -6 & 2 \\ -6 & 4 & -4 \\ 2 & -4 & 0 \end{bmatrix}$

Apply $R_2 \to 5R_2 + 6R_1$ and $R_3 \to 5R_3 - 2R_1$

$$A - 3I \sim \begin{bmatrix} 5 & -6 & 2 \\ 0 & -16 & -8 \\ 0 & -8 & -4 \end{bmatrix}$$

Apply $R_2 \to \dfrac{R_2}{-8}$ and $R_3 \to \dfrac{R_3}{-4}$

$$A-3I \sim \begin{bmatrix} 5 & -6 & 2 \\ 0 & 2 & 1 \\ 0 & 2 & 1 \end{bmatrix}$$

Apply $R_3 \to R_3 - R_2$

$$A-3I \sim \begin{bmatrix} 5 & -6 & 2 \\ 0 & 2 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

Take $x_3 = k$, where k is a constant

Now $2x_2 + x_3 = 0 \Rightarrow x_2 = -\dfrac{k}{2}$

Also $5x_1 - 6x_2 + 2x_3 = 0 \Rightarrow x_1 = -k$

$\Rightarrow \begin{bmatrix} -k \\ -\dfrac{k}{2} \\ k \end{bmatrix}$ is a general solution of $(A-\lambda_2 I)X = O$

Choose the eigen vector $X_2 = \begin{bmatrix} -2 \\ -1 \\ 2 \end{bmatrix}$ corresponding to $\lambda_2 = 3$.

To find eigen vectors corresponding to $\lambda_3 = 15$, consider $(A-\lambda_3 I)X = O$ then

$$A-15I = \begin{bmatrix} -7 & -6 & 2 \\ -6 & -8 & -4 \\ 2 & -4 & -12 \end{bmatrix}$$

Apply $R_2 \to \dfrac{R_2}{2}$ and $R_3 \to \dfrac{R_3}{2}$

$$A-15I \sim \begin{bmatrix} -7 & -6 & 2 \\ -3 & -4 & -2 \\ 1 & -2 & -6 \end{bmatrix}$$

Apply $R_2 \to 7R_2 - 3R_1$ and $R_3 \to 7R_3 + R_1$

$$A - 15I \sim \begin{bmatrix} -7 & -6 & 2 \\ 0 & -10 & -20 \\ 0 & -20 & -40 \end{bmatrix}$$

Apply $R_2 \to \dfrac{R_2}{10}$ and $R_3 \to \dfrac{R_3}{20}$

$$A - 15I \sim \begin{bmatrix} -7 & -6 & 2 \\ 0 & -1 & -2 \\ 0 & -1 & -2 \end{bmatrix}$$

Apply $R_3 \to R_3 - R_2$

$$A - 15I \sim \begin{bmatrix} -7 & -6 & 2 \\ 0 & -1 & -2 \\ 0 & 0 & 0 \end{bmatrix}$$

Take $x_3 = k$, where k is a constant

Now $-x_2 - 2x_3 = 0 \Rightarrow x_2 = -2k$

Also $-7x_1 - 6x_2 + 2x_3 = 0 \Rightarrow x_1 = 2k$

$\Rightarrow \begin{bmatrix} 2k \\ -2k \\ k \end{bmatrix}$ is a general solution of $(A - \lambda_3 I)X = O$.

Choose the eigen vector $X_3 = \begin{bmatrix} 2 \\ -2 \\ 1 \end{bmatrix}$ corresponding to $\lambda_3 = 15$.

Since the eigen vectors $X_1 = \begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix}$, $X_2 = \begin{bmatrix} -2 \\ -1 \\ 2 \end{bmatrix}$ and $X_3 = \begin{bmatrix} 2 \\ -2 \\ 1 \end{bmatrix}$ are corresponding to distinct eigen values of a symmetric matrix A, they are linearly independent and orthogonal. The modal matrix is

$$P = \begin{bmatrix} 1 & -2 & 2 \\ 2 & -1 & -2 \\ 2 & 2 & 1 \end{bmatrix}$$

$\Rightarrow P$ is an orthogonal matrix

$$\Rightarrow D = P^{-1}AP = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 15 \end{bmatrix}$$

The canonical form or normal form of the given quadratic form is

$\lambda_1 x_1^{\ 2} + \lambda_2 x_2^{\ 2} + \lambda_3 x_3^{\ 2}$, where $\lambda_1 = 0, \lambda_2 = 3$ and $\lambda_3 = 15$

The rank of A is $r = 2$. The index of the given quadratic form is $s = 2$ and the signature is $2s - r = 2$. The given quadratic form is positive semidefinite.

Problem: Find the nature of the quadratic form $x_1^2 + 4x_2^2 + x_3^2 - 4x_1x_2 + 2x_1x_3 - 4x_2x_3$.

Solution: Given quadratic form is

$$X^T AX = x_1^2 + 4x_2^2 + x_3^2 - 4x_1x_2 + 2x_1x_3 - 4x_2x_3$$

Comparing with

$X^T AX = a_{11}x_1^2 + a_{22}x_2^2 + a_{33}x_3^2 + 2a_{12}x_1x_2 + 2a_{23}x_2x_3 + 2a_{13}x_1x_3$, we get

$a_{11} = 1,\ a_{22} = 4,\ a_{33} = 1,\ a_{12} = a_{21} = -2,\ a_{13} = a_{31} = 1$ and $a_{23} = a_{32} = -2$

Now the symmetric matrix corresponding to the given quadratic form is

$$A = \begin{bmatrix} 1 & -2 & 1 \\ -2 & 4 & -2 \\ 1 & -2 & 1 \end{bmatrix}$$

$$\Rightarrow A - \lambda I = \begin{bmatrix} 1-\lambda & -2 & 1 \\ -2 & 4-\lambda & -2 \\ 1 & -2 & 1-\lambda \end{bmatrix}$$

The characteristic equation of A is $\det(A - \lambda I) = 0$

$$\Rightarrow \lambda^3 - 6\lambda^2 = 0$$

Hence $\lambda_1 = 0, \lambda_2 = 0$ and $\lambda_3 = 6$ are the eigen values of A.

The given quadratic form is positive semidefinite.

Exercise

1. Write a symmetric matrix corresponding to the following quadratic forms
   a. $ax^2 + by^2 + 2hxy$
   b. $x^2 + 2y^2 + 3z^2 + 4xy + 5yz + 6xz$
   c. $x_1^2 + 2x_2^2 - 7x_3{}^2 - 4x_1x_2 + 8x_1x_3$
   d. $2xy + 6xz - 4yz$
   e. $x_1^2 + 6x_1x_2 + 5x_2^2$
2. Find a quadratic form corresponding to the following matrices
   a. $\begin{bmatrix} a & h & g \\ h & b & f \\ g & f & c \end{bmatrix}$
   b. $\begin{bmatrix} 0 & 5 & -1 \\ 5 & 1 & 6 \\ -1 & 6 & 2 \end{bmatrix}$
   c. $\begin{bmatrix} 1 & 2 & 3 \\ 2 & 0 & 3 \\ 3 & 3 & 1 \end{bmatrix}$
   d. $\begin{bmatrix} 2 & 1 & 5 \\ 1 & 3 & -2 \\ 5 & -2 & 4 \end{bmatrix}$
   e. $\begin{bmatrix} 1 & 5 & 7 \\ 5 & 4 & 6 \\ 7 & 6 & 3 \end{bmatrix}$
3. Reduce the following quadratic forms to normal form
   a. $6x_1^2 + 3x_2^2 + 3x_3^2 - 4x_1x_2 - 2x_2x_3 + 4x_1x_3$
   b. $4x^2 + 3y^2 + z^2 - xy - 6yz + 4xz$

c. $3x^2 + 2y^2 + 3z^2 - 2xy - 2yz$

d. $3x^2 + 5y^2 + 3z^2 - 2xy - 2yz + 2xz$

e. $x_1^2 + 3x_2^2 + 3x_3^2 - 2x_2x_3$

4. Find rank, signature and index of the following quadratic forms

a. $3x^2 + 2y^2 + 3z^2 - 2xy - 2yz$

b. $6x^2 - 4xy + 2y^2$

c. $3x_1^2 + 5x_2^2 + 3x_3^2 - 2x_1x_2 - 2x_2x_3 + 2x_1x_3$

d. $2x^2 + 2y^2 + 2z^2 - 2xy - 2yz - 2xz$

e. $x^2 + y^2 + 4xy$

5. Find the nature of the following quadratic forms

a. $2x^2 + 2y^2 + 2z^2 + 2yz$

b. $x^2 + 2y^2 + 3z^2 + 4xy + 5yz + 6xz$

c. $x_1^2 + 2x_2^2 - 7x_3{}^2 - 4x_1x_2 + 8x_1x_3$

d. $x^2 - y^2 + 4z^2 + 4xy + 2yz + 6xz$

e. $x_1^2 + 6x_1x_2 + 5x_2^2$

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