Matrices

Definition: A *matrix* is a rectangular arrangement of elements in rows and columns. If a matrix A has m rows and n columns then we say that the *order* of the matrix is $m \times n$. If A is a matrix of order $m \times n$ then it can be written as

$$A = \left[a_{ij}\right]_{m\times n} = \begin{bmatrix} a_{11} & a_{12} & a_{13} & \dots & a_{1n} \\ a_{21} & a_{22} & a_{23} & \dots & a_{2n} \\ \dots & & & & \\ \dots & & & & \\ a_{m1} & a_{m2} & a_{m3} & \dots & a_{mn} \end{bmatrix}_{m\times n}$$

Definition: If A is a matrix of order $m \times n$, then

1. A is called a *rectangular matrix* if $m \neq n$.
2. A is called a *square matrix* if $m = n$.
3. A is called a *row matrix* if $m = 1$.
4. A is called a *column matrix* if $n = 1$.

Definition: If all the entries of a matrix A are real numbers then it is called a *real matrix.* If at least one entry of a matrix A is a complex number then A is called a *complex matrix.*

Definition: A matrix $\left[a_{ij}\right]_{m\times n}$ is called the zero matrix if $a_{ij} = 0$ for all i and j. The zero matrix is denoted by the symbol O.

Definition: If $A = \left[a_{ij}\right]_{m\times n}$ is a real or complex matrix of order $m \times n$ and if k is a constant then

$$ka = \left[ka_{ij}\right]_{m\times n} = \begin{bmatrix} ka_{11} & ka_{12} & ka_{13} & \dots & ka_{1n} \\ ka_{21} & ka_{22} & ka_{23} & \dots & ka_{2n} \\ \dots & & & & \\ \dots & & & & \\ ka_{m1} & ka_{m2} & ka_{m3} & \dots & ka_{mn} \end{bmatrix}_{m\times n}$$

Definition: If $A=\left[a_{ij}\right]_{m\times n}$ and $B=\left[b_{ij}\right]_{m\times n}$ are any two real or complex matrices of same order then the sum $A+B$ is also an $m\times n$ matrix, given by

$$A+B=\left[a_{ij}+b_{ij}\right] = \begin{bmatrix} a_{11}+b_{11} & a_{12}+b_{12} & \dots & a_{1n}+b_{1n} \\ a_{21}+b_{21} & a_{22}+b_{22} & \dots & a_{2n}+b_{2n} \\ \dots & & & \\ \dots & & & \\ a_{m1}+b_{m1} & a_{m2}+b_{m2} & \dots & a_{mn}+b_{mn} \end{bmatrix}_{m\times n}$$

Remarks: Let $M_{m\times n}(\mathbb{R})$ be the set of all real matrices of order $m\times n$. Then $M_{m\times n}(\mathbb{R})$ satisfies the following properties with respect to the addition of matrices

1. If $A\in M_{m\times n}(\mathbb{R})$ and $B\in M_{m\times n}(\mathbb{R})$ then $A+B\in M_{m\times n}(\mathbb{R})$. That is if A and B are any two real matrices of order $m\times n$ then $A+B$ is also a real matrix of the same order $m\times n$. (*Closure property*)
2. If A, B *and* C are any three real matrices of order $m\times n$ then $(A+B)+C=A+(B+C)$ (*Associative law*)
3. If A is a real matrix of order $m\times n$ then $A+O=O+A=A$, where O is the zero matrix of order $m\times n$. (*Existence of Identity*)
4. If $A=\left[a_{ij}\right]_{m\times n}$ is a real matrix of order $m\times n$ then $-A=\left[-a_{ij}\right]_{m\times n}$ is also a real matrix of order $m\times n$ and $A+(-A)=(-A)+A=O$. The matrix $-A$ is called the additive inverse of A. (*Existence of Inverse*)
5. If A and B are any two real matrices of order $m\times n$ then
   $A+B=B+A$.
   (Commutative law)

Definition: If $A=\left[a_{ij}\right]_{m\times k}$ and $B=\left[b_{ij}\right]_{k\times n}$ are any two real or complex matrices then the *multiplication* of A and B is defined to be the matrix $AB=\left[c_{ij}\right]_{m\times n}$ where

$$c_{ij}=a_{i1}b_{1j}+a_{i2}b_{2j}+a_{i3}b_{3j}+\dots+a_{ik}b_{kj}\,.$$

Definition: If $A=\left[a_{ij}\right]_{m\times n}$ is any real or complex matrix of order $m\times n$ then the *transpose* matrix of A is defined by

$$A^T=\begin{bmatrix} a_{11} & a_{21} & a_{31} & \dots & a_{m1} \\ a_{12} & a_{22} & a_{32} & \dots & a_{m2} \\ \dots \\ \dots \\ a_{1n} & a_{2n} & a_{3n} & \dots & a_{nm} \end{bmatrix}_{n\times m}$$

Remarks:

1. If A and B are any two matrices of the same order then $\left(A+B\right)^T=A^T+B^T$
2. If k is a constant and A is a matrix of order $m\times n$ then $\left(k\,A\right)^T=k\,A^T$.
3. If A is a matrix of order $m\times k$ and B is a matrix of order $k\times n$ then $\left(AB\right)^T=B^TA^T$
4. If A is a matrix of order $m\times n$ then $\left(A^T\right)^T=A$

Definition: If A is a square matrix of order $n\times n$ such that

$$A=\left[a_{ij}\right]_{n\times n}=\begin{bmatrix} a_{11} & a_{12} & a_{13} & \dots & a_{1n} \\ a_{21} & a_{22} & a_{23} & \dots & a_{2n} \\ \dots \\ \dots \\ a_{n1} & a_{n2} & a_{n3} & \dots & a_{nn} \end{bmatrix}_{n\times n}$$

Then the elements $a_{11},a_{22},a_{33}....\ and\ a_{nn}$ are called *principal diagonal* entries.

The *trace* of A is defined to be the number $tr(A)=a_{11}+a_{22}+...+a_{nn}$.

Remarks: If A and B are any two square matrices of same order then

1. $tr(A+B)=tr(A)+tr(B)$
2. $tr(kA)=k\,tr(A)$ where k is a constant
3. $tr(A)=tr\left(A^T\right)$

Definition: A real square matrix A is said to be

1. *Symmetric* if $A = A^T$
2. *Skew - symmetric* if $A = -A^T$ or $A + A^T = O$ where O is the zero matrix
3. *Orthogonal* if $AA^T = I$ where I is the identity matrix given by $I = \left[e_{ij}\right]_{n\times n}$,

   $e_{ij} = \begin{cases} 1 & if \;\; i = j \\ 0 & if \;\; i \neq j \end{cases}$

Theorem: Every real square matrix can be expressed as a sum of a symmetric matrix and a skew– symmetric matrix in a unique way.

Proof: Let A be a real square matrix. Let $P = \frac{1}{2}\left[A + A^T\right]$ and $Q = \frac{1}{2}\left[A - A^T\right]$

Then $P + Q = \frac{1}{2}\left[A + A^T\right] + \frac{1}{2}\left[A - A^T\right]$

$$= \frac{1}{2}\left[A + A^T + A - A^T\right]$$

$$= A$$

$\Rightarrow P + Q = A$

Now we prove that P is symmetric and Q is skew - symmetric

We have $P^T = \left(\frac{1}{2}\left(A + A^T\right)\right)^T = \frac{1}{2}\left(A + A^T\right)^T = \frac{1}{2}\left[A^T + \left(A^T\right)^T\right] = \frac{1}{2}\left(A^T + A\right) = P$

$\Rightarrow P^T = P$

$\Rightarrow$ P is symmetric

And also,

$$Q^T = \left(\frac{1}{2}\left(A - A^T\right)\right)^T = \frac{1}{2}\left(A - A^T\right)^T = \frac{1}{2}\left[A^T - \left(A^T\right)^T\right] = \frac{1}{2}\left(A^T - A\right) = -\frac{1}{2}\left(A - A^T\right) = -Q$$

$\Rightarrow$ $Q^T = -Q$

Hence $A=P+Q$, where $P=\frac{1}{2}\left[A+A^T\right]$ is symmetric and $Q=\frac{1}{2}\left[A-A^T\right]$ is skew - symmetric

Now we prove that this representation is unique. Suppose that $A=R+S$ where R is symmetric and S is skew – symmetric

Since R is symmetric, $R^T=R$

Since S is skew – symmetric, $S^T=-S$

Now $P=\frac{1}{2}\left[A+A^T\right]=\frac{1}{2}\left[(R+S)+(R+S)^T\right]=\frac{1}{2}\left(R+S+R^T+S^T\right)=\frac{1}{2}(R+S+R-S)$

$=R$

$\Rightarrow P=R$

Similarly, $Q=\frac{1}{2}\left[A-A^T\right]=\frac{1}{2}\left[(R+S)-(R+S)^T\right]=\frac{1}{2}\left(R+S-\left(R^T+S^T\right)\right)$

$$=\frac{1}{2}\left(R+S-(R-S)\right)$$

$$=\frac{1}{2}(R+S-R+S)$$

$$=S$$

$\Rightarrow Q=S$

Thus every real square matrix can be written as a sum of a symmetric matrix and a skew – symmetric matrix in a unique way.

Remarks:

1. If A and B are any two symmetric matrices then $A+B$ and $A-B$ are also symmetric
2. If A and B are any two skew- symmetric matrices then $A+B$ and $A-B$ are also skew – symmetric
3. If k is a positive real number and if A is symmetric then kA is also symmetric

4. If k is a positive real number and if A is skew - symmetric then kA is also skew – symmetric
5. If A and B are orthogonal matrices then AB is also an orthogonal matrix

Problem: Resolve the matrix $A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 6 & -3 & 0 \end{bmatrix}$ into a symmetric matrix and a skew – symmetric matrix.

Solution: Given $A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 6 & -3 & 0 \end{bmatrix}$

$$\Rightarrow A^T = \begin{bmatrix} 1 & 4 & 6 \\ 2 & 5 & -3 \\ 3 & 6 & 0 \end{bmatrix}$$

Then $P = \frac{1}{2}\left[A + A^T\right] = \frac{1}{2}\begin{bmatrix} 2 & 6 & 9 \\ 6 & 10 & 3 \\ 9 & 3 & 0 \end{bmatrix}$ and $Q = \frac{1}{2}\left[A - A^T\right] = \frac{1}{2}\begin{bmatrix} 0 & -2 & -3 \\ 2 & 0 & 9 \\ 3 & -9 & 0 \end{bmatrix}$

Clearly $A = P + Q$, where P is symmetric and Q is skew – symmetric. The matrix P is called the symmetric part of A and Q is called the skew – symmetric part of A.

Definition: If $A = \left[a_{ij}\right]_{m\times n}$ is a complex matrix then its *conjugate matrix* $\overline{A}$ is defined by $\overline{A} = \left[\overline{a_{ij}}\right]_{m\times n}$ where $\overline{a_{ij}}$ denotes the conjugate of a_{ij}. The *conjugate transpose* of $A = \left[a_{ij}\right]_{m\times n}$ is defined by $\overline{A^T} = \left[\overline{a_{ij}}\right]^T_{m\times n}$.

Remarks: If A and B are any two complex matrices then

1. $\overline{[A+B]^T} = \overline{A^T} + \overline{B^T}$
2. $\overline{[k\,A]^T} = \overline{k}\,\overline{A^T}$ where k is a real or complex number
3. $\overline{\left[\overline{A}\right]} = A$

Definition: A complex square matrix A is said to be

1. Hermitian if $\overline{A^T} = A$
2. *Skew-Hermitian* if $\overline{A^T} = -A$
3. *Unitary* if $A\,\overline{A^T} = I$ where I is the identity matrix

Theorem: Every complex square matrix can be expressed as a sum of a Hermitian matrix and a Skew-Hermitian matrix in a unique way.

Proof: Let A be a complex square matrix. Let $P = \frac{1}{2}\left[A + \overline{A^T}\right]$ and $Q = \frac{1}{2}\left[A - \overline{A^T}\right]$

Then $P + Q = \frac{1}{2}\left[A + \overline{A^T}\right] + \frac{1}{2}\left[A - \overline{A^T}\right]$

$$= \frac{1}{2}\left[A + \overline{A^T} + A - \overline{A^T}\right]$$

$$= A$$

$\Rightarrow P + Q = A$

Now we prove that P is Hermitian and Q is Skew-Hermitian

Consider $P = \frac{1}{2}\left[A + \overline{A^T}\right]$

$$\Rightarrow \quad P^T = \left(\frac{1}{2}\left(A + \overline{A^T}\right)\right)^T = \frac{1}{2}\left[A^T + \overline{\left(A^T\right)^T}\right] = \frac{1}{2}\left(A^T + \overline{A}\right)$$

$$\Rightarrow \quad \overline{P^T} = \overline{\frac{1}{2}\left(A^T + \overline{A}\right)} = \frac{1}{2}\left(\overline{A^T} + \overline{\overline{A}}\right) = \frac{1}{2}\left(\overline{A^T} + A\right) = P$$

$$\Rightarrow \quad \overline{P^T} = P$$

$\Rightarrow P$ is Hermitian

Now $Q = \frac{1}{2}\left[A - \overline{A^T}\right]$

$$\Rightarrow \quad Q^T = \left(\frac{1}{2}\left(A - \overline{A^T}\right)\right)^T = \frac{1}{2}\left[A^T - \overline{\left(A^T\right)^T}\right] = \frac{1}{2}\left(A^T - \overline{A}\right)$$

$\Rightarrow \overline{Q^T} = \overline{\frac{1}{2}\left(A^T - \overline{A}\right)} = \frac{1}{2}\left(\overline{A^T} - \overline{\overline{A}}\right) = \frac{1}{2}\left(\overline{A^T} - A\right) = -\frac{1}{2}\left(A - \overline{A^T}\right) = -Q$

$\Rightarrow \overline{Q^T} = -Q$

$\Rightarrow Q$ is skew- Hermitian

Hence $A = P + Q$, where $P = \frac{1}{2}\left[A + \overline{A^T}\right]$ is Hermitian and $Q = \frac{1}{2}\left[A - \overline{A^T}\right]$ is Skew-Hermitian

Now we prove that this representation is unique. Suppose that $A = R + S$ where R is Hermitian and S is Skew-Hermitian

Since R is Hermitian, $\overline{R^T} = R$

Since S is Skew-Hermitian, $\overline{S^T} = -S$

Now $P = \frac{1}{2}\left[A + \overline{A^T}\right] = \frac{1}{2}\left[(R+S) + \overline{(R+S)^T}\right] = \frac{1}{2}\left(R + S + \overline{R^T} + \overline{S^T}\right)$

$= \frac{1}{2}(R + S + R - S) = R$

$\Rightarrow P = R$

Similarly, $Q = \frac{1}{2}\left[A - \overline{A^T}\right] = \frac{1}{2}\left[(R+S) - \overline{(R+S)^T}\right] = \frac{1}{2}\left(R + S - \left(\overline{R^T} + \overline{S^T}\right)\right)$

$= \frac{1}{2}\left(R + S - (R - S)\right)$

$= \frac{1}{2}(R + S - R + S)$

$= S$

$\Rightarrow Q = S$

Thus every complex square matrix can be written as a sum of a Hermitian matrix and a Skew-Hermitian matrix in a unique way.

Definition: If $A=\left[a_{ij}\right]_{n\times n}$ is a square matrix then its *determinant* is defined as $\det(A)=a_{11}C_{11}+a_{12}C_{12}+a_{13}C_{13}+\ldots+a_{1n}C_{1n}$ where C_{jk} is called the *co-factor* of the element a_{jk} and it is given by $C_{jk}=(-1)^{j+k}M_{jk}$ and M_{jk} is called the *minor* of a_{jk}. The minor M_{jk} of a_{jk} is the determinant of the $(n-1)\times(n-1)$ matrix that survives where the row and column containing a_{jk} are struck out.

Definition: A square matrix A is said to be

1. *Singular* if $\det(A)=0$
2. *Non – singular* or invertible, if $\det(A)\neq 0$.

Definition: If $A=\left[a_{ij}\right]_{n\times n}$ is a square matrix of order n then it said to be

1. *Upper triangular* if $a_{ij}=0$ for all $i>j$
2. *Lower triangular* if $a_{ij}=0$ for all $i<j$
3. *Diagonal* if $a_{ij}=0$ for all $i\neq j$

Remarks:

1. If any row or columns of A is modified by adding α times the corresponding elements of another row or column to it, yielding a new matrix B, then $\det(B)=\det(A)$.
2. If any rows or columns of A are interchanged, yielding a new matrix B, then $\det(B)=-\det(A)$
3. If A is triangular or diagonal, then $\det(A)$ is simply the product of the diagonal elements, $\det(A)=a_{11}\,a_{22}\,a_{33}\ldots a_{nn}$
4. If all the elements of any row or column are zero, then $\det(A)=0$
5. $\det(\alpha A)=\alpha^{n}\det(A)$
6. $\det\left(A^{T}\right)=\det(A)$
7. $\det(A+B)\neq\det(A)+\det(B)$
8. $\det(A\,B)=\det(A)\,\det(B)$

Definition: If A is any square matrix then the *adjoint* of A is defined to be the transpose of the cofactor matrix of A. That is $adj(A) = \left[C_{jk}\right]^T$ where $C_{jk} = (-1)^{j+k} M_{jk}$ and M_{jk} is called the *minor* of a_{jk}. The minor M_{jk} of a_{jk} is the determinant of the $(n-1)\times(n-1)$ matrix that survives where the row and column containing a_{jk} are struck out.

Definition: If A is a non-singular matrix then its multiplicative inverse A^{-1} is defined by $A^{-1} = \dfrac{1}{\det A} adj(A)$. Clearly, $A A^{-1} = A^{-1} A = I$ where I is the identity matrix.

Remark:

Let A be any square matrix of order $n \times n$. Suppose that A is non-singular to find the inverse A^{-1} of A, we use the Gauss – Jordan method. In this method, we write $A = I_n A$ where I_n is the identity matrix of order $n \times n$. Then we apply row operations to A on the LHS and to the matrix I_n on the RHS until the equation becomes $I_n = BA$ then the matrix B is the inverse of A i.e., $B = A^{-1}$. This method is useful if the order of A is large.

Exercise

1. Resolve the following matrices as a sum of symmetric and skew-symmetric parts.

   a. $\begin{bmatrix} 1 & 2 & -3 \\ 4 & 2 & 6 \\ 6 & -3 & 0 \end{bmatrix}$ b. $\begin{bmatrix} 6 & 3 & 1 \\ 2 & 4 & -8 \\ 3 & -6 & 1 \end{bmatrix}$ c. $\begin{bmatrix} -1 & -2 & -1 \\ 6 & 12 & 6 \\ 5 & 10 & 4 \end{bmatrix}$ d. $\begin{bmatrix} -1 & -1 & 1 \\ 2 & 2 & -2 \\ -3 & -3 & 3 \end{bmatrix}$

2. Find the determinant of the following matrices.

   a. $\begin{bmatrix} 1 & -1 & -5 \\ -1 & -1 & 4 \\ -3 & -10 & 4 \end{bmatrix}$ b. $\begin{bmatrix} 1 & -1 & -3 \\ -1 & -1 & -10 \\ -5 & 4 & 4 \end{bmatrix}$ c. $\begin{bmatrix} 4 & -4 & 3 \\ 9 & -10 & 1 \\ 3 & 5 & -18 \end{bmatrix}$ d. $\begin{bmatrix} 1 & 4 & 3 \\ 2 & 0 & 3 \\ 6 & 8 & 9 \end{bmatrix}$

3. Find the inverse of the following matrices by finding their adjoints.

a. $\begin{bmatrix} 1 & a & b \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ b. $\begin{bmatrix} cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}$ c. $\begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}$

4. Find the inverse of the following matrices by using elementary transformations.

a. $\begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 2 \end{bmatrix}$ b. $\begin{bmatrix} 1 & 1 & 1 & 1 \\ -2 & 1 & -1 & -2 \\ -4 & -2 & -3 & 1 \\ 1 & 1 & 1 & 0 \end{bmatrix}$ c. $\begin{bmatrix} 1 & -0 & 0 & 0 & 0 \\ -2 & -1 & -1 & 3 & 2 \\ 3 & -4 & 2 & 3 & -1 \\ -4 & -12 & -1 & 9 & -3 \\ 2 & 0 & 0 & 3 & 1 \end{bmatrix}$

d. $\begin{bmatrix} 2 & -6 & -2 & -3 \\ 5 & -13 & -4 & -7 \\ -1 & 4 & 1 & 2 \\ 0 & 1 & 0 & 1 \end{bmatrix}$

5. Verify whether the following matrices are orthogonal or not.

a. $\begin{bmatrix} 0 & 2b & c \\ a & b & -c \\ a & -b & c \end{bmatrix}$ b. $\begin{bmatrix} -1 & 2 & 2 \\ 2 & -1 & 2 \\ 2 & 2 & -1 \end{bmatrix}$ c. $\begin{bmatrix} \cos\theta & 0 & \sin\theta \\ 0 & 1 & 0 \\ \sin\theta & 0 & \cos\theta \end{bmatrix}$ d. $\begin{bmatrix} \frac{-2}{3} & \frac{1}{3} & \frac{2}{3} \\ \frac{2}{3} & \frac{2}{3} & \frac{1}{3} \\ \frac{1}{3} & \frac{-2}{3} & \frac{2}{3} \end{bmatrix}$

6. Express the following complex matrices as a sum of Hermitian and skew-Hermitian matrices

a. $\begin{bmatrix} 3 & 7-4i & -2+5i \\ 7+4i & -2 & 3+i \\ -2-5i & 3-i & 4 \end{bmatrix}$ b. $\begin{bmatrix} 1+i & 2 & 5-5i \\ 2i & 2+i & 4+2i \\ -1+i & -4 & 7 \end{bmatrix}$

c. $\begin{bmatrix} 2+3i & 1-i & 2+i \\ -2i & 4 & 2i \\ -4i & -4i & i \end{bmatrix}$ d. $\begin{bmatrix} 2 & 3+2i & 2+i \\ -2i & 4 & 2i \\ -4i & -4i & i \end{bmatrix}$

7. Show that $A = \begin{bmatrix} -i & 3+2i & -2-i \\ -3+2i & 0 & 3-4i \\ 2-i & -3-4i & -2i \end{bmatrix}$ is skew- Hermitian.

8. If $A=\begin{bmatrix} 3 & 7-4i & -2+5i \\ 7+4i & -2 & 3+i \\ -2-5i & 3-i & 4 \end{bmatrix}$ then show that A is Hermitian and iA is skew- Hermitian.

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Rank of a Matrix

Definition: We say that the *rank* of a matrix A is r if

1. There exists a at least one non-zero minor of order r.
2. All the minors of order $r+1$ and above are zero.

The rank of a matrix A is denoted by $\rho(A)$ or $r(A)$.

Remarks:

1. If A is a rectangular matrix of order $m\times n$ then $\rho(A)\le \min(m,n)$.
2. If A is any matrix then $\rho(A) = \rho(A^T)$.
3. The rank of a zero matrix is zero.
4. The rank of a non – zero matrix is positive.
5. If A is a square matrix of order $n\times n$ and $\det(A)\neq 0$ then $\rho(A)=n$.
6. If A is a square matrix of order $n\times n$ and $\det(A)=0$ then $\rho(A)=n$.
7. Elementary row operations do not alter the rank of a matrix.

Echelon Form:

A matrix is said to be in Echelon from if it has the following properties.

1. The zero rows must follow non- zero rows
2. The number of zero rows before the first non-zero element of a row is less than the number of such zeros in the next row.

The rank of a matrix A is the number of non-zero rows in an Echelon form of A .

Problem: Find the rank of $A=\begin{bmatrix} 2 & 1 & -3 & 4 \\ 2 & 4 & -2 & 5 \\ 0 & 3 & 1 & 3 \\ 2 & 1 & -3 & -2 \end{bmatrix}$ using elementary row operations.

Solution: Given matrix is $A = \begin{bmatrix} 2 & 1 & -3 & 4 \\ 2 & 4 & -2 & 5 \\ 0 & 3 & 1 & 3 \\ 2 & 1 & -3 & -2 \end{bmatrix}$

Applying $R_2 \to R_2 - R_1$ and $R_4 \to R_4 - R_1$. Then we get

$$A \sim \begin{bmatrix} 2 & 1 & -3 & 4 \\ 0 & 3 & 1 & 1 \\ 0 & 3 & 1 & 3 \\ 0 & 0 & 0 & -6 \end{bmatrix}$$

Applying $R_2 \to R_3 - R_2$. Then we obtain

$$A \sim \begin{bmatrix} 2 & 1 & -3 & 4 \\ 0 & 3 & 1 & 1 \\ 0 & 0 & 0 & 2 \\ 0 & 0 & 0 & -6 \end{bmatrix}$$

Applying $R_2 \to R_4 + 3R_3$. Then we obtain

$$A \sim \begin{bmatrix} 2 & 1 & -3 & 4 \\ 0 & 3 & 1 & 1 \\ 0 & 0 & 0 & 2 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

The number of non-zero rows in the echelon form of A is 3.

Hence the rank of A is $\rho(A) = 3$.

Problem: Find the rank of the matrix $A = \begin{bmatrix} 1 & 4 & 3 & -2 & 1 \\ -2 & -3 & -1 & 4 & 3 \\ -1 & 6 & 7 & 2 & 9 \\ -3 & 3 & 6 & 6 & 12 \end{bmatrix}$ using elementary row operations.

Solution: Given matrix is $A = \begin{bmatrix} 1 & 4 & 3 & -2 & 1 \\ -2 & -3 & -1 & 4 & 3 \\ -1 & 6 & 7 & 2 & 9 \\ -3 & 3 & 6 & 6 & 12 \end{bmatrix}$

Applying $R_2 \to R_2 + 2R_1$, $R_3 \to R_3 + R_1$ and $R_4 \to R_4 + 3R_1$

$$A \sim \begin{bmatrix} 1 & 4 & 3 & -2 & 1 \\ 0 & 5 & 5 & 0 & 5 \\ 0 & 10 & 10 & 0 & 10 \\ 0 & 15 & 15 & 0 & 15 \end{bmatrix}$$

Applying $R_2 \to \frac{R_2}{5}$, $R_3 \to \frac{R_3}{5}$ and $R_4 \to \frac{R_4}{5}$

$$A \sim \begin{bmatrix} 1 & 4 & 3 & -2 & 1 \\ 0 & 1 & 1 & 0 & 1 \\ 0 & 1 & 1 & 0 & 1 \\ 0 & 1 & 1 & 0 & 1 \end{bmatrix}$$

Applying $R_3 \to R_3 - R_2$ and $R_4 \to R_4 - R_2$

$$A \sim \begin{bmatrix} 1 & 4 & 3 & -2 & 1 \\ 0 & 1 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

The number of non-zero rows in the echelon form of A is 2.

Hence the rank of A is $\rho(A) = 2$.

Normal form:

A normal form of a matrix A is given by $\begin{bmatrix} I_r & 0 \\ 0 & 0 \end{bmatrix}$ where r is some positive integer. If a matrix A is converted by using elementary operations into the normal form $\begin{bmatrix} I_r & 0 \\ 0 & 0 \end{bmatrix}$ then the rank of A is $\rho(A) = r$.

Problem:

Find the rank of a matrix $A = \begin{bmatrix} 1 & 3 & 4 & 5 \\ 1 & 2 & 6 & 7 \\ 1 & 5 & 0 & 10 \end{bmatrix}$ by using normal form.

Solution: Given matrix is $A = \begin{bmatrix} 1 & 3 & 4 & 5 \\ 1 & 2 & 6 & 7 \\ 1 & 5 & 0 & 10 \end{bmatrix}_{3\times 4}$

Apply $R_2 \to R_2 - R_1$ and $R_3 \to R_3 - R_1$

$$A \sim \begin{bmatrix} 1 & 3 & 4 & 5 \\ 0 & -1 & 2 & 2 \\ 0 & 2 & -4 & 5 \end{bmatrix}$$

Apply $R_3 \to R_3 + 2R_1$

$$A \sim \begin{bmatrix} 1 & 3 & 4 & 5 \\ 0 & -1 & 2 & 2 \\ 0 & 0 & 0 & 9 \end{bmatrix}$$

Apply $C_2 \to C_2 - 3C_1, \quad C_3 \to C_3 - 4C_1 \quad and \quad C_4 \to C_4 - 5C_1$

$$A \sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 2 & 2 \\ 0 & 0 & 0 & 9 \end{bmatrix}$$

Apply $C_3 \to C_3 + 2C_2 \quad and \quad C_4 \to C_4 + 2C_2$

$$A \sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 9 \end{bmatrix}$$

Apply $R_2 \to \dfrac{R_2}{-1} \quad and \quad R_3 \to \dfrac{R_3}{9}$

$$A \sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Apply $C_3 \leftrightarrow C_4$

$$A \sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} \sim [I_3 \quad 0]$$

$\Rightarrow$ The rank of A is $\rho(A) = 3$.

Procedure: To find two non-singular matrices P and Q in such a way that the matrix PAQ is in the normal form, we adapt the following procedure.

1. Suppose that A is a rectangular matrix of order $m \times n$
2. Write $A = I_m \, A \, I_n$ where I_m and I_n are identity matrices of orders $m \times m$ and $n \times n$ respectively
3. Apply row operations to A on the LHS and to I_m on the RHS
4. Apply column operations to A on the LHS and to I_n on the RHS
5. Convert the matrix A on the LHS of $A = I_m \, A \, I_n$ to normal form. Then $A = I_m \, A \, I_n$ takes the form $PAQ = N(A)$, where $N(A)$ is the normal form of A. Observe that the matrices P and Q are non- singular.

Remark: If A is a non- singular matrix then we can find two non- singular matrices P and Q such that PAQ is in the normal form. Then $A^{-1} = QP$.

Problem: Obtain two non-singular matrices P and Q such that PAQ is in the normal form, where $A = \begin{bmatrix} 1 & 1 & 2 \\ 1 & 2 & 3 \\ 0 & -1 & -1 \end{bmatrix}$ here $m = n = 3$.

Solution: Given matrix is $A = \begin{bmatrix} 1 & 1 & 2 \\ 1 & 2 & 3 \\ 0 & -1 & -1 \end{bmatrix}$

Write $A = I_m \, A \, I_n \Rightarrow I_3 \, A \, I_3$

$$\Rightarrow \begin{bmatrix} 1 & 1 & 2 \\ 1 & 2 & 3 \\ 0 & -1 & -1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} A \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Apply $R_2 \to R_2 - R_1$

$$\Rightarrow \begin{bmatrix} 1 & 1 & 2 \\ 0 & 1 & 1 \\ 0 & -1 & -1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} A \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Apply $R_3 \to R_3 + R_2$

$$\Rightarrow \begin{bmatrix} 1 & 1 & 2 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ -1 & 1 & 1 \end{bmatrix} A \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Apply $C_2 \to C_2 - C_1 \quad and \quad C_3 \to C_3 - 2C_1$

$$\Rightarrow \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ -1 & 1 & 1 \end{bmatrix} A \begin{bmatrix} 1 & -1 & -2 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Apply $C_3 \to C_3 - C_2$

$$\Rightarrow \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ -1 & 1 & 1 \end{bmatrix} A \begin{bmatrix} 1 & -1 & -1 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} I_2 & 0 \\ 0 & 0 \end{bmatrix} = P\,A\,Q \quad \text{where } P = \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ -1 & 1 & 1 \end{bmatrix} \text{ and } Q = \begin{bmatrix} 1 & -1 & -1 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{bmatrix}$$

Here P and Q are non- singular matrices and PAQ is in the normal form. The rank of A is $\rho(A) = 2$.

Exercise

1. Find the Rank of the following matrices by using Echelon form.

a. $\begin{bmatrix} 1 & 2 & 3 & 0 \\ 2 & -3 & -1 & 4 \\ 3 & 2 & 1 & 3 \\ 6 & 8 & 7 & 5 \end{bmatrix}$ b. $\begin{bmatrix} -1 & -3 & 3 & -1 \\ 1 & 1 & -1 & 0 \\ 2 & -5 & 2 & -3 \\ -1 & 1 & 0 & 1 \end{bmatrix}$

c. $\begin{bmatrix} 2 & 1 & 3 & 5 \\ 4 & 2 & 1 & 3 \\ 8 & 4 & 7 & 13 \\ 8 & 4 & -3 & -1 \end{bmatrix}$ d. $\begin{bmatrix} 1 & 2 & 3 & 4 \\ 1 & 2 & 3 & 1 \\ -1 & -1 & 1 & 2 \\ -1 & -1 & 1 & -1 \end{bmatrix}$

e. $\begin{bmatrix} 8 & 1 & 3 & 6 \\ 0 & 3 & 2 & 2 \\ -8 & -1 & -3 & 4 \end{bmatrix}$

2. Find the Rank of the following matrices by reducing into normal form.

a. $\begin{bmatrix} 2 & -2 & 0 & 6 \\ 4 & 2 & 0 & 2 \\ 1 & -1 & 0 & 3 \\ 1 & -2 & 1 & 2 \end{bmatrix}$ b. $\begin{bmatrix} 1 & 2 & 1 & 2 \\ 1 & 3 & 2 & 2 \\ 2 & 4 & 3 & 4 \\ 3 & 7 & 5 & 6 \end{bmatrix}$

c. $\begin{bmatrix} 4 & 3 & 2 & 1 \\ 5 & 1 & -1 & 2 \\ 0 & 1 & 2 & 3 \\ 1 & -1 & 3 & -2 \end{bmatrix}$ d. $\begin{bmatrix} 1 & 2 & 3 & 4 \\ 2 & 1 & 4 & 3 \\ 3 & 0 & 5 & -10 \end{bmatrix}$

e. $\begin{bmatrix} 5 & 3 & 14 & 4 \\ 0 & 1 & 2 & 1 \\ 1 & -1 & 2 & 0 \end{bmatrix}$

3. Obtain two non – singular matrices P and Q such that PAQ is in the Normal Form for the following matrices

a. $\begin{bmatrix} 3 & 2 & -1 & 5 \\ 5 & 1 & 4 & -2 \\ 1 & -4 & 11 & -19 \end{bmatrix}$ b. $\begin{bmatrix} 1 & -2 & 3 & 4 \\ -2 & 4 & -1 & -3 \\ -1 & 2 & 7 & 6 \end{bmatrix}$

c. $\begin{bmatrix} 1 & -1 & -1 \\ 1 & 1 & 1 \\ 3 & 1 & 1 \end{bmatrix}$ d. $\begin{bmatrix} 2 & -1 & 3 \\ 1 & 1 & 1 \\ 1 & -1 & 1 \end{bmatrix}$

d. $\begin{bmatrix} 1 & 1 & 3 \\ 1 & 3 & -3 \\ -2 & -4 & -4 \end{bmatrix}$

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System of linear equations

A system of m liner equations in n unknowns can be written as

$$a_{11}x_1 + a_{12}x_2 + a_{13}x_3 + \ldots + a_{12}x_n = b_1$$

$$a_{21}x_1 + a_{22}x_2 + a_{23}x_3 + \ldots + a_{2n}x_n = b_2$$

$$a_{31}x_1 + a_{32}x_2 + a_{33}x_3 + \ldots + a_{3n}x_n = b_3$$

$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

$$a_{m1}x_1 + a_{m2}x_2 + a_{m3}x_3 + \ldots + a_{mn}x_n = b_m$$

We restrict the values of m and n to be finite and a_{ij}, b_i to be real numbers. This system of m liner equations in n unknowns can be represented as a matrix equation $AX = B$, where

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} & \ldots a_{1n} \\ a_{21} & a_{22} & a_{23} & \ldots a_{2n} \\ a_{31} & a_{32} & a_{33} & \ldots a_{3n} \\ \ldots & \ldots & \ldots & \ldots \\ a_{m1} & a_{m2} & a_{m3} & \ldots a_{mn} \end{bmatrix}_{m\times n} \quad X = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ . \\ . \\ . \\ x_n \end{bmatrix}_{n\times 1} \quad \text{and} \quad B = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \\ . \\ . \\ . \\ b_m \end{bmatrix}_{m\times 1}$$

A value of $X = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ . \\ . \\ . \\ x_n \end{bmatrix}_{n\times 1}$ which satisfies the matrix equation $AX = B$ is called a solution of the given system of m liner equations in n unknowns. If a system of linear equations $AX = B$ has a solution then it is said to be *consistent*. If a system $AX = B$ of linear equations has no solution the it is said to be *inconsistent*. The augmented matrix for the given system of m liner equations with n unknowns is given by

$$[A/B] = A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \dots a_{1n} & b_1 \\ a_{21} & a_{22} & a_{23} \dots a_{2n} & b_2 \\ a_{31} & a_{32} & a_{33} \dots a_{3n} & b_3 \\ \dots\dots\dots\dots & & & \\ a_{m1} & a_{m2} & a_{m3} \dots a_{mn} & b_m \end{bmatrix}_{m\times(n+1)}$$

1. If $rank\ A \neq rank\,[A/B]$ then the given system of linear equations has no solution or the system is inconsistent.
2. If $rank\ A = rank\,[A/B]$ then the given system of linear equations has atleast one solution or the system is consistent.
3. If $rank\ A = rank\,[A/B] = n$, where n is the number of unknowns then the system of linear equations has a unique solution or the system has exactly one solution.
4. If $rank\ A = rank\,[A/B] < n$, where n is the number of unknowns then the system of linear equations has infinitely many solutions.
5. If the number of equations is less than the number of unknowns in a system and if the system is consistent then it cannot have a unique solution.

Definition: A system $AX = B$ of m liner equations in n unknowns is said to be a *homogeneous system* if $B = O$ where the O is the zero matrix.

If B is a non-zero matrix then the system $AX = B$ is said to be a *non- homogeneous system.*

Remarks:

1. A homogeneous system $AX = O$ is always consistent.
2. The zero matrix O is always a solution for $AX = O$ and it is called the trivial solution.
3. If A is a non-singular matrix then the system $AX = O$ has exactly one solution and it is the trivial solution $X = O$ only.
4. If A is singular matrix then the system $AX = O$ has infinitely many solutions.

Gauss elimination method

Consider a system $AX = B$ of m liner equations in n unknowns.

1. Verify whether the given system of linear equations is consistent or not
2. If the given system of linear equations is consistent then convert the augmented matrix $[A/B]$ as an upper triangular matrix by using elementary row operations
3. First find x_n and by using the value of x_n find x_{n-1} thus find all the unknowns by the method of substitution.

Problem: Verify whether the following system of linear equations is consistent or not.

$$\begin{aligned}&2x_1 + 3x_2 - 2x_3 = 4\\&x_1 - 2x_2 + x_3 = 3\\&7x_1 - x_3 = 2\end{aligned}$$

Solution: The given system of linear equations is

$$\begin{aligned}&2x_1 + 3x_2 - 2x_3 = 4\\&x_1 - 2x_2 + x_3 = 3\\&7x_1 - x_3 = 2\end{aligned}$$

This system can be represented as $AX = B$, where

$$A = \begin{bmatrix} 2 & 3 & -2 \\ 1 & -2 & 1 \\ 7 & 0 & -1 \end{bmatrix} \quad X = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \quad B = \begin{bmatrix} 4 \\ 3 \\ 2 \end{bmatrix}$$

The augmented matrix is $[A/B] = \begin{bmatrix} 2 & 3 & -2 & 4 \\ 1 & -2 & 1 & 3 \\ 7 & 0 & -1 & 2 \end{bmatrix}$

Consider

$$A = \begin{bmatrix} 2 & 3 & -2 \\ 1 & -2 & 1 \\ 7 & 0 & -1 \end{bmatrix} \text{ and } [A/B] = \begin{bmatrix} 2 & 3 & -2 & 4 \\ 1 & -2 & 1 & 3 \\ 7 & 0 & -1 & 2 \end{bmatrix}$$

Apply $R_2 \to 2R_2 - R_1$ and $R_3 \to 2R_3 - 7R_1$

$$A \sim \begin{bmatrix} 2 & 3 & -2 \\ 0 & -7 & 4 \\ 0 & -21 & 12 \end{bmatrix} \text{ and } [A/B] \sim \begin{bmatrix} 2 & 3 & -2 & 4 \\ 0 & -7 & 4 & 2 \\ 0 & -21 & 12 & -24 \end{bmatrix}$$

Apply $R_3 \to R_3 - 3R_2$

$$A \sim \begin{bmatrix} 2 & 3 & -2 \\ 0 & -7 & 4 \\ 0 & 0 & 0 \end{bmatrix} \text{ and } [A/B] \sim \begin{bmatrix} 2 & 3 & -2 & 4 \\ 0 & -7 & 4 & 2 \\ 0 & 0 & 0 & -30 \end{bmatrix}$$

$\Rightarrow$ $rank\ A = 2$ and $rank\,[A/B] = 3$

$\Rightarrow$ $rank\ A \neq rank\,[A/B]$

$\Rightarrow$ The given system is inconsistent.

Problem: Verify whether the following system of linear equations is consistent or not.

$$\begin{aligned} x + 2y - z &= 3 \\ 3x - y + 2z &= 1 \\ 2x - 2y + 3z &= 2 \\ x - y + z &= -1 \end{aligned}$$

Solution: The given system of linear equations is

$$x+2y-z=3$$
$$3x-y+2z=1$$
$$2x-2y+3z=2$$
$$x-y+z=-1$$

This system can be represented as $AX=B$, where

$$A=\begin{bmatrix}1 & 2 & -1\\ 3 & -1 & 2\\ 2 & -2 & 3\\ 1 & -1 & 1\end{bmatrix} \quad X=\begin{bmatrix}x\\ y\\ z\end{bmatrix} \quad B=\begin{bmatrix}3\\ 1\\ 2\\ -1\end{bmatrix}$$

The augmented matrix is $[A/B]=\begin{bmatrix}1 & 2 & -1 & 3\\ 3 & -1 & 2 & 1\\ 2 & -2 & 3 & 2\\ 1 & -1 & 1 & -1\end{bmatrix}$

Consider

$$A=\begin{bmatrix}1 & 2 & -1\\ 3 & -1 & 2\\ 2 & -2 & 3\\ 1 & -1 & 1\end{bmatrix} \quad \text{and} \quad [A/B]=\begin{bmatrix}1 & 2 & -1 & 3\\ 3 & -1 & 2 & 1\\ 2 & -2 & 3 & 2\\ 1 & -1 & 1 & -1\end{bmatrix}$$

Apply $R_2 \to R_2-3R_1$; $R_3 \to R_3-2R_1$ and $R_4 \to R_4-R_1$

$$A \sim \begin{bmatrix}1 & 2 & -1\\ 0 & -7 & 5\\ 0 & -6 & 5\\ 0 & -3 & 2\end{bmatrix} \quad \text{and} \quad [A/B] \sim \begin{bmatrix}1 & 2 & -1 & 3\\ 0 & -7 & 5 & -8\\ 0 & -6 & 5 & -4\\ 0 & -3 & 2 & -4\end{bmatrix}$$

Apply $R_3 \to 7R_3-6R_2$ and $R_4 \to 7R_4-3R_2$

$$A \sim \begin{bmatrix}1 & 2 & -1\\ 0 & -7 & 5\\ 0 & 0 & 5\\ 0 & 0 & -1\end{bmatrix} \quad \text{and} \quad [A/B] \sim \begin{bmatrix}1 & 2 & -1 & 3\\ 0 & -7 & 5 & -8\\ 0 & 0 & 5 & 20\\ 0 & 0 & -1 & -4\end{bmatrix}$$

Apply $R_4 \to 5R_4+R_3$

$$A \sim \begin{bmatrix} 1 & 2 & -1 \\ 0 & -7 & 5 \\ 0 & 0 & 5 \\ 0 & 0 & 0 \end{bmatrix} \quad \text{and} \quad [A/B] \sim \begin{bmatrix} 1 & 2 & -1 & 3 \\ 0 & -7 & 5 & -8 \\ 0 & 0 & 5 & 20 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$\Rightarrow$ $rank\ A = 3$ and $rank\ [A/B] = 3$

$\Rightarrow$ $rank\ A = rank\ [A/B]$

$\Rightarrow$ The given system is consistent and $rank\ A = rank\ [A/B] = 3$, the number of unknowns.

$\Rightarrow$ The given system has a unique solution

Problem: Solve the following system of linear equations

$$\begin{aligned} x_1 + x_2 - x_3 &= 1 \\ 3x_1 + x_2 + x_3 &= 9 \\ x_1 - x_2 + 4x_3 &= 8 \end{aligned}$$

Solution: The given system of linear equations is

$$\begin{aligned} x_1 + x_2 - x_3 &= 1 \\ 3x_1 + x_2 + x_3 &= 9 \\ x_1 - x_2 + 4x_3 &= 8 \end{aligned}$$

This system can be represented as $AX = B$, where

$$A = \begin{bmatrix} 1 & 1 & -1 \\ 3 & 1 & 1 \\ 1 & -1 & 4 \end{bmatrix} \quad X = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \quad B = \begin{bmatrix} 1 \\ 9 \\ 8 \end{bmatrix}$$

The augmented matrix is $[A/B] = \begin{bmatrix} 1 & 1 & -1 & 1 \\ 3 & 1 & 1 & 9 \\ 1 & -1 & 4 & 8 \end{bmatrix}$

Consider

$$A = \begin{bmatrix} 1 & 1 & -1 \\ 3 & 1 & 1 \\ 1 & -1 & 4 \end{bmatrix} \quad \text{and} \quad [A/B] = \begin{bmatrix} 1 & 1 & -1 & 1 \\ 3 & 1 & 1 & 9 \\ 1 & -1 & 4 & 8 \end{bmatrix}$$

Apply $R_2 \to R_2 - 3R_1$ and $R_3 \to R_3 - R_1$

$$A \sim \begin{bmatrix} 1 & 1 & -1 \\ 0 & -2 & 4 \\ 0 & -2 & 5 \end{bmatrix} \quad \text{and} \quad [A/B] \sim \begin{bmatrix} 1 & 1 & -1 & 1 \\ 0 & -2 & 4 & 6 \\ 0 & -2 & 5 & 7 \end{bmatrix}$$

Apply $R_3 \to R_3 - R_2$

$$A \sim \begin{bmatrix} 1 & 1 & -1 \\ 0 & -2 & 4 \\ 0 & 0 & 1 \end{bmatrix} \quad \text{and} \quad [A/B] \sim \begin{bmatrix} 1 & 1 & -1 & 1 \\ 0 & -2 & 4 & 6 \\ 0 & 0 & 1 & 1 \end{bmatrix}$$

$\Rightarrow$ $rank\ A = 3$ and $rank\,[A/B] = 3$

$\Rightarrow$ $rank\ A = rank\,[A/B]$

$\Rightarrow$ The given system is consistent and $rank\ A = rank\,[A/B] = 3$, the number of unknowns.

$\Rightarrow$ The given system has a unique solution

Now consider $[A/B] \sim \begin{bmatrix} 1 & 1 & -1 & 1 \\ 0 & -2 & 4 & 6 \\ 0 & 0 & 1 & 1 \end{bmatrix}$

By using the last row of $[A/B]$, we get $x_3 = 1$

By using the second row of $[A/B]$, we get

$$-2x_2 + 4x_3 = 6 \Rightarrow x_2 - 2x_3 = -3$$

$$\Rightarrow x_2 - 2 = -3$$

$$\Rightarrow x_2 = -1$$

By using the first row of $[A/B]$, we get

$$x_1 + x_2 - x_3 = 1 \quad \Rightarrow x_1 - 1 - 1 = 1 \quad \Rightarrow x_1 = 3$$

$$\Rightarrow X = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 3 \\ -1 \\ 1 \end{bmatrix}$$ is a unique solution of the given system.

Problem: Solve the following system of linear equations

$$5x + 3y + 7z = 4\,;\, 3x + 26y + 2z = 9\,;\, 7x + 2y + 10z = 5$$

Solution: The given system of linear equations is

$$5x + 3y + 7z = 4\,;\, 3x + 26y + 2z = 9\,;\, 7x + 2y + 10z = 5$$

This system can be represented as $AX = B$, where

$$A = \begin{bmatrix} 5 & 3 & 7 \\ 3 & 26 & 2 \\ 7 & 2 & 10 \end{bmatrix} \quad X = \begin{bmatrix} x \\ y \\ z \end{bmatrix} \quad B = \begin{bmatrix} 4 \\ 9 \\ 5 \end{bmatrix}$$

The augmented matrix is $[A/B] = \begin{bmatrix} 5 & 3 & 7 & 4 \\ 3 & 26 & 2 & 9 \\ 7 & 2 & 10 & 5 \end{bmatrix}$

Consider

$$A = \begin{bmatrix} 5 & 3 & 7 \\ 3 & 26 & 2 \\ 7 & 2 & 10 \end{bmatrix} \text{ and } [A/B] = \begin{bmatrix} 5 & 3 & 7 & 4 \\ 3 & 26 & 2 & 9 \\ 7 & 2 & 10 & 5 \end{bmatrix}$$

Apply $R_2 \to 5R_2 - 3R_1$ and $R_3 \to 5R_3 - 7R_1$

$$A \sim \begin{bmatrix} 5 & 3 & 7 \\ 0 & 121 & -11 \\ 0 & -11 & 1 \end{bmatrix} \text{ and } [A/B] \sim \begin{bmatrix} 5 & 3 & 7 & 4 \\ 0 & 121 & -11 & 33 \\ 0 & -11 & 1 & -3 \end{bmatrix}$$

Apply $R_2 \to \dfrac{R_2}{11}$

$$A \sim \begin{bmatrix} 5 & 3 & 7 \\ 0 & 11 & -1 \\ 0 & -11 & 1 \end{bmatrix} \quad \text{and} \quad [A/B] \sim \begin{bmatrix} 5 & 3 & 7 & 4 \\ 0 & 11 & -1 & 3 \\ 0 & -11 & 1 & -3 \end{bmatrix}$$

Apply $R_3 \to R_3 + R_2$

$$A \sim \begin{bmatrix} 5 & 3 & 7 \\ 0 & 11 & -1 \\ 0 & 0 & 0 \end{bmatrix} \quad \text{and} \quad [A/B] \sim \begin{bmatrix} 5 & 3 & 7 & 4 \\ 0 & 11 & -1 & 3 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$\Rightarrow$ $rank\ A = 2$ and $rank\,[A/B] = 2$

$\Rightarrow$ $rank\ A = rank\,[A/B]$

$\Rightarrow$ The given system is consistent and $rank\ A = rank\,[A/B] = 2 < 3$, the number of unknowns

$\Rightarrow$ The given system of linear equations has infinitely many solutions

Consider

$$[A/B] \sim \begin{bmatrix} 5 & 3 & 7 & 4 \\ 0 & 11 & -1 & 3 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Put $z = k$, where k is a constant.

From the second row of $[A/B]$,

$11y - z = 3 \Rightarrow 11y - k = 3$

$\Rightarrow 11y = 3 + k$

$\Rightarrow y = \dfrac{3+k}{11}$

From the first row of $[A/B]$,

$5x + 3y + 7z = 4 \Rightarrow 5x + 3\left(\dfrac{3+k}{11}\right) + 7k = 4$

$$\Rightarrow \; x = \frac{7-16k}{11}$$

$$\Rightarrow X = \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} \frac{7-16k}{11} \\ \\ \frac{3+k}{11} \\ \\ k \end{bmatrix}$$ is a general solution of the given system of linear equations.

Put $k = 8$. Then $X_1 = \begin{bmatrix} -11 \\ 1 \\ 8 \end{bmatrix}$ is a solution of the given system of liner equations.

Problem: Solve the following homogeneous system of linear equations

$$\begin{aligned} x + 3y - 2z &= 0 \\ 2x - y + 4z &= 0 \\ x - 11y + 14z &= 0 \end{aligned}$$

Solution: The given system of linear equations is

$$\begin{aligned} x + 3y - 2z &= 0 \\ 2x - y + 4z &= 0 \\ x - 11y + 14z &= 0 \end{aligned}$$

This system can be represented as $AX = O$, where

$$A = \begin{bmatrix} 1 & 3 & -2 \\ 2 & -1 & 4 \\ 1 & -11 & 14 \end{bmatrix} \quad X = \begin{bmatrix} x \\ y \\ z \end{bmatrix} \quad O = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Consider

$$A = \begin{bmatrix} 1 & 3 & -2 \\ 2 & -1 & 4 \\ 1 & -11 & 14 \end{bmatrix}$$

Apply $R_2 \to R_2 - 2R_1$ and $R_3 \to R_3 - R_1$

$$A \sim \begin{bmatrix} 1 & 3 & -2 \\ 0 & -7 & 8 \\ 0 & -14 & 16 \end{bmatrix}$$

Apply $R_3 \to \dfrac{R_3}{2}$

$$A \sim \begin{bmatrix} 1 & 3 & -2 \\ 0 & -7 & 8 \\ 0 & -7 & 8 \end{bmatrix}$$

Apply $R_3 \to R_3 - R_2$

$$A \sim \begin{bmatrix} 1 & 3 & -2 \\ 0 & -7 & 8 \\ 0 & 0 & 0 \end{bmatrix}$$

$\Rightarrow$ $rank\ A = 2 < 3$, the number of unknowns

$\Rightarrow$ the given homogeneous system has infinitely many solutions

Consider $A \sim \begin{bmatrix} 1 & 3 & -2 \\ 0 & -7 & 8 \\ 0 & 0 & 0 \end{bmatrix}$

Put $z = k$, where k is a constant.

From the second row of A, we have $-7y + 8z = 0 \Rightarrow -7y + 8k = 0 \Rightarrow y = \dfrac{8k}{7}$

From the first row of A, we have $x + 3y - 2z = 0 \Rightarrow x + \dfrac{24k}{7} - 2k = 0 \Rightarrow x = \dfrac{10k}{7}$

$\Rightarrow X = \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} \dfrac{10k}{7} \\ \dfrac{8k}{7} \\ k \end{bmatrix}$ is a general solution of the given system of linear

equations.

Put $k = 7$. Then $X_1 = \begin{bmatrix} 7 \\ 8 \\ 7 \end{bmatrix}$ is a solution of the given homogeneous system of linear equations.

Gauss – Seidel iteration method

Consider the following system of linear equations

$$a_{11}x_1 + a_{12}x_2 + a_{13}x_3 = b_1$$

$$a_{21}x_1 + a_{22}x_2 + a_{23}x_3 = b_2$$

$$a_{31}x_1 + a_{32}x_2 + a_{33}x_3 = b_3$$

This system of linear equations is said to be *diagonally dominant* , if

$|a_{11}| \geq |a_{12}| + |a_{13}|$; $|a_{22}| \geq |a_{21}| + |a_{23}|$ and $|a_{33}| \geq |a_{31}| + |a_{32}|$

Suppose that the given system of linear equations is diagonally dominant.
Consider the given system

$$a_{11}x_1 + a_{12}x_2 + a_{13}x_3 = b_1$$

$$a_{21}x_1 + a_{22}x_2 + a_{23}x_3 = b_2$$

$$a_{31}x_1 + a_{32}x_2 + a_{33}x_3 = b_3$$

$$\Rightarrow x_1 = \frac{b_1}{a_{11}} - \frac{a_{12}}{a_{11}}x_2 - \frac{a_{13}}{a_{11}}x_3 \quad \rightarrow (1)$$

$$\Rightarrow x_2 = \frac{b_2}{a_{22}} - \frac{a_{21}}{a_{22}}x_1 - \frac{a_{23}}{a_{22}}x_3 \quad \rightarrow (2)$$

$$\Rightarrow x_3 = \frac{b_3}{a_{33}} - \frac{a_{31}}{a_{33}}x_1 - \frac{a_{32}}{a_{33}}x_2 \quad \rightarrow (3)$$

Now we choose initial approximations $x_2^{(0)}$ and $x_3^{(0)}$ for the values of x_2 and x_3. By substituting these initial approximations in eq (1) and we find $x_1^{(1)}$ as follows

$$x_1^{(1)} = \frac{b_1}{a_{11}} - \frac{a_{12}}{a_{11}} x_2^{(0)} - \frac{a_{13}}{a_{11}} x_3^{(0)}$$

By using the values of $x_1^{(1)}$ and $x_3^{(0)}$, we find $x_2^{(1)}$ as follows

$$x_2^{(1)} = \frac{b_2}{a_{22}} - \frac{a_{21}}{a_{22}} x_1^{(1)} - \frac{a_{23}}{a_{22}} x_3^{(0)}$$

By using $x_1^{(1)}$ and $x_2^{(1)}$, we find $x_3^{(1)}$ as follows

$$x_3^{(1)} = \frac{b_3}{a_{33}} - \frac{a_{31}}{a_{33}} x_1^{(1)} - \frac{a_{32}}{a_{33}} x_2^{(1)}$$

Now we get the first approximation $X_1 = \begin{bmatrix} x_1^{(1)} \\ x_2^{(1)} \\ x_3^{(1)} \end{bmatrix}$ where

$$x_1^{(1)} = \frac{b_1}{a_{11}} - \frac{a_{12}}{a_{11}} x_2^{(0)} - \frac{a_{13}}{a_{11}} x_3^{(0)} \quad \rightarrow (4)$$

$$x_2^{(1)} = \frac{b_2}{a_{22}} - \frac{a_{21}}{a_{22}} x_1^{(1)} - \frac{a_{23}}{a_{22}} x_3^{(0)} \quad \rightarrow (5)$$

$$x_3^{(1)} = \frac{b_3}{a_{33}} - \frac{a_{31}}{a_{33}} x_1^{(1)} - \frac{a_{32}}{a_{33}} x_2^{(1)} \quad \rightarrow (6)$$

Next we find the second approximation $X_2 = \begin{bmatrix} x_1^{(2)} \\ x_2^{(2)} \\ x_3^{(2)} \end{bmatrix}$ to a solution of the given system of linear equations, where

$$x_1^{(2)} = \frac{b_1}{a_{11}} - \frac{a_{12}}{a_{11}} x_2^{(1)} - \frac{a_{13}}{a_{11}} x_3^{(1)} \quad \rightarrow (7)$$

$$x_2^{(2)} = \frac{b_2}{a_{22}} - \frac{a_{21}}{a_{22}} x_1^{(2)} - \frac{a_{23}}{a_{22}} x_3^{(1)} \quad \rightarrow (8)$$

$$x_3^{(2)} = \frac{b_3}{a_{33}} - \frac{a_{31}}{a_{33}} x_1^{(2)} - \frac{a_{32}}{a_{33}} x_2^{(2)} \quad \rightarrow (9)$$

The third approximation $X_3 = \begin{bmatrix} x_1^{(3)} \\ x_2^{(3)} \\ x_3^{(3)} \end{bmatrix}$ can be obtained by the following

$$x_1^{(3)} = \frac{b_1}{a_{11}} - \frac{a_{12}}{a_{11}} x_2^{(2)} - \frac{a_{13}}{a_{11}} x_3^{(2)} \quad \rightarrow (10)$$

$$x_2^{(3)} = \frac{b_2}{a_{22}} - \frac{a_{21}}{a_{22}} x_1^{(3)} - \frac{a_{23}}{a_{22}} x_3^{(2)} \quad \rightarrow (11)$$

$$x_3^{(3)} = \frac{b_3}{a_{33}} - \frac{a_{31}}{a_{33}} x_1^{(3)} - \frac{a_{32}}{a_{33}} x_2^{(3)} \quad \rightarrow (12)$$

We continue this process and we get a sequence of approximations $X_1 = \begin{bmatrix} x_1^{(1)} \\ x_2^{(1)} \\ x_3^{(1)} \end{bmatrix}$,

$$X_2 = \begin{bmatrix} x_1^{(2)} \\ x_2^{(2)} \\ x_3^{(2)} \end{bmatrix}, \quad X_3 = \begin{bmatrix} x_1^{(3)} \\ x_2^{(3)} \\ x_3^{(3)} \end{bmatrix}, \ldots.$$

From this sequence of approximations we take the desired solution of the given system of linear equations.

Problem: Find the first three approximations to a solution of the following system of linear equations by using Gauss- Seidel iteration method

$$x_1 - \frac{1}{4}x_2 - \frac{1}{4}x_3 = \frac{1}{2}$$

$$-\frac{1}{4}x_1 + x_2 - \frac{1}{4}x_4 = \frac{1}{2}$$

$$-\frac{1}{4}x_1 + x_3 - \frac{1}{4}x_4 = \frac{1}{4}$$

$$-\frac{1}{4}x_2 - \frac{1}{4}x_3 + x_4 = \frac{1}{4}$$

Solution: Given system of linear equations is

$$x_1 - \frac{1}{4}x_2 - \frac{1}{4}x_3 = \frac{1}{2}$$

$$-\frac{1}{4}x_1 + x_2 - \frac{1}{4}x_4 = \frac{1}{2}$$

$$-\frac{1}{4}x_1 + x_3 - \frac{1}{4}x_4 = \frac{1}{4}$$

$$-\frac{1}{4}x_2 - \frac{1}{4}x_3 + x_4 = \frac{1}{4}$$

$\Rightarrow x_1 = 0.5 + 0.25x_2 + 0.25x_3 \quad \rightarrow \quad (1)$

$x_2 = 0.5 + 0.25x_1 + 0.25x_4 \quad \rightarrow \quad (2)$

$x_3 = 0.25 + 0.25x_1 + 0.25x_4 \quad \rightarrow \quad (3)$

$x_4 = 0.25 + 0.25x_2 + 0.25x_3 \quad \rightarrow \quad (4)$

By taking $x_2^{(0)} = x_3^{(0)} = x_4^{(0)} = 0$ as an initial approximation, we get

$x_1^{(1)} = 0.5 + 0.25x_2^{(0)} + 0.25x_3^{(0)} \Rightarrow x_1^{(1)} = 0.5$

By using $x_1^{(1)} = 0.5$ and $x_4^{(0)} = 0$ we get $x_2^{(1)} = 0.5 + 0.25x_1^{(1)} + 0.25x_4^{(0)}$

$\Rightarrow x_2^{(1)} = 0.5 + 0.25x_1^{(1)} \Rightarrow x_2^{(1)} = 0.625$

By using $x_1^{(1)} = 0.5$ and $x_4^{(0)} = 0$, we get

$x_3^{(1)} = 0.25 + 0.25x_1^{(1)} + 0.25x_4^{(0)}$

$\Rightarrow x_3^{(1)} = 0.25 + 0.25x_1^{(1)} \Rightarrow x_3^{(1)} = 0.375$

By using $x_2^{(1)} = 0.625$ and $x_3^{(1)} = 0.375$, we get

$x_4^{(1)} = 0.25 + 0.25x_2^{(1)} + 0.25x_3^{(1)} \Rightarrow x_4^{(1)} = 0.5$

Then the first approximation is given by $X_1 = \begin{bmatrix} x_1^{(1)} \\ x_2^{(1)} \\ x_3^{(1)} \\ x_4^{(1)} \end{bmatrix} = \begin{bmatrix} 0.5 \\ 0.625 \\ 0.375 \\ 0.5 \end{bmatrix}$

The second approximation $X_2 = \begin{bmatrix} x_1^{(2)} \\ x_2^{(2)} \\ x_3^{(2)} \\ x_4^{(2)} \end{bmatrix}$ is obtained as follows

$$x_1^{(2)} = 0.5 + 0.25x_2^{(1)} + 0.25x_3^{(1)}$$

$$x_2^{(2)} = 0.5 + 0.25x_1^{(2)} + 0.25x_4^{(1)}$$

$$x_3^{(2)} = 0.25 + 0.25x_1^{(2)} + 0.25x_4^{(1)}$$

$$x_4^{(2)} = 0.25 + 0.25x_2^{(2)} + 0.25x_3^{(2)}$$

$$\Rightarrow X_2 = \begin{bmatrix} x_1^{(2)} \\ x_2^{(2)} \\ x_3^{(2)} \\ x_4^{(2)} \end{bmatrix} = \begin{bmatrix} 0.75 \\ 0.812 \\ 0.562 \\ 0.593 \end{bmatrix}$$

The third approximation $X_3 = \begin{bmatrix} x_1^{(3)} \\ x_2^{(3)} \\ x_3^{(3)} \\ x_4^{(3)} \end{bmatrix}$ is obtained as follows

$$\Rightarrow X_3 = \begin{bmatrix} x_1^{(3)} \\ x_2^{(3)} \\ x_3^{(3)} \\ x_4^{(3)} \end{bmatrix} = \begin{bmatrix} 0.843 \\ 0.859 \\ 0.609 \\ 0.617 \end{bmatrix}$$

Exercise

1. Verify whether the following systems are consistent or inconsistent

a. $\begin{aligned} 5x+3y+14z&=4\\ y+2z&=1\\ x+y+2z&=0\\ 2x+y+6z&=2 \end{aligned}$ b. $\begin{aligned} x+y+z+w&=4\\ x+y+z-w&=2\\ x-y+z-w&=0 \end{aligned}$ c. $\begin{aligned} x-4y+7z&=8\\ 3x+8y-2z&=6\\ 7x-8y+26z&=3 \end{aligned}$

d. $\begin{aligned} x-4y+7z&=14\\ 3x+8y-2z&=13\\ 7x-8y+26z&=5 \end{aligned}$ e. $\begin{aligned} x+2y+z&=14\\ 3x+4y+z&=11\\ 2x+3y+z&=11 \end{aligned}$ f. $\begin{aligned} x+y+z&=1\\ x-y+2z&=1\\ x-y+2z&=5\\ 2x-2y+3z&=1 \end{aligned}$

2. Verify the consistency and hence solve the following systems of linear equations

a. $\begin{aligned} 2x+y+z&=10\\ 3x+y-z&=6\\ x-2y-4z&=-10 \end{aligned}$ b. $\begin{aligned} x+2y+3z&=4\\ 5x+6y+7z&=8\\ 9x+10y+11z&=12 \end{aligned}$ c. $\begin{aligned} x_1+x_3&=1\\ x_1+2x_2-x_3-2x_4&=5\\ x_1-x_2+2x_3+x_4&=0\\ 2x_1+x_2+x_3-x_4&=4 \end{aligned}$

d. $\begin{aligned} x_1-x_2+2x_3+x_4&=-1\\ 2x_1+x_2+x_3-x_4&=4\\ x_1+2x_2-x_3+2x_4&=5\\ x_1+x_3&=1 \end{aligned}$ e. $\begin{aligned} x+y+z&=7\\ 3x+3y+4z&=24\\ 2x+y+3z&=16 \end{aligned}$

3. Solve the following homogeneous systems of linear equations

a. $\begin{aligned} x+3y+2z&=0\\ 2x-y+3z&=0\\ 3x-5y+4z&=0\\ x+17y+4z&=0 \end{aligned}$ b. $\begin{aligned} 3x+3y-z+5t&=0\\ 3x-y+2z-7t&=0\\ 4x-y-3z+6t&=0\\ x-2y+4z-7t&=0 \end{aligned}$ c. $\begin{aligned} x+y-2z+3w&=0\\ x-2y+z-w&=0\\ 4x+y-5z+w&=0\\ 5x-7y+2z-w&=0 \end{aligned}$

d. $$\begin{aligned} 2x - y + 3z &= 0 \\ 3x + 2y + z &= 0 \\ x - 4y + 5z &= 0 \end{aligned}$$

e. $$\begin{aligned} x + y - 3z + 2w &= 0 \\ 2x - y + 2z - 3w &= 0 \\ 3x - 2y + z - 4w &= 0 \\ -4x + y - 3z + w &= 0 \end{aligned}$$

4. Find the first four approximations to a solution of the following systems of linear equations by using Gauss-Seidel Method

a. $$\begin{aligned} 2x_1 - x_2 &= 7 \\ -x_1 + 2x_2 - x_3 &= 1 \\ -x_2 + 2x_3 &= 1 \end{aligned}$$

b. $$\begin{aligned} 20x + y - 2z &= 17 \\ 3x + 20y - z &= -18 \\ 2x - 3y + 20z &= 25 \end{aligned}$$

c. $$\begin{aligned} 10x + y + z &= 12 \\ 2x + 10y + z &= 13 \\ 2x + 2y + 10z &= 14 \end{aligned}$$

d. $$\begin{aligned} 20x + 2y + 6z &= 28 \\ x + 20y + 9z &= -23 \\ 2x - 7y - 20z &= -57 \end{aligned}$$